Advanced and algorithmic graph theory

Summer term 2022

Exercise sheet 3

Exercises for the exercise session on 09/05/2022

(Bonus problems are not counted towards the total number of problems, but solving a bonus problem will earn you a bonus point.)

Problem 3.1. Let G be bipartite with sides A and B and suppose that each vertex v has a preference order $\leq_v$ on its set of neighbours. A stable matching M is called *A-optimal* if each vertex $a \in A$ is matched to its "best" neighbour among all vertices that are possible partners for a in stable matchings. (Formally: If $ab \in M$ and $ab' \in M'$ for some stable matching M', then $b \leq_a b'$.) We define *B-optimal*, *A-pessimal* (worst possible for A) and *B-pessimal* analogously.
Prove that a stable matching is A-optimal if and only if it is B-pessimal. Furthermore, prove that every bipartite G has a unique A-optimal stable matching.

Problem 3.2. Show that the stable matching generated by STABLE is A-optimal (and thus also B-pessimal).

Problem 3.3. Prove that Tutte's Theorem (Theorem 2.8 from the lecture) implies Hall's Theorem (Theorem 2.2) and that Theorem 2.9 by Gallai and Edmonds implies König's Theorem (Theorem 2.1).
Hint. For the first part, start by showing that the case $|A| = |B|$ of Hall's theorem implies the general case. Then prove that in this case, the marriage condition implies the Tutte condition.

Problem 3.4. Prove Proposition 3.4 from the lecture: The following statements are equivalent for every plane graph G on $n \geq 3$ vertices.

(i) G is maximally planar;

(ii) G is maximally plane;

(iii) G is a triangulation;

(iv) $\|G\| = 3n - 6$.

Problem 3.5. Prove Corollary 3.8 from the lecture: Every maximally planar graph G with at least four vertices is 3-connected.
Hint. Suppose, for contradiction, that $\{u, v\}$ is a separator of size two. If $uv \in E(G)$, derive a contradiction to the maximality of G. Otherwise, use the non-planarity of $G + uv$ to obtain a contradiction to the planarity of G.

Problem 3.6. A graph is called *outerplanar* if it is planar and has a drawing in which all vertices lie on the boundary of the outer face. Prove that the following statements are equivalent for a graph G.

(i) G is outerplanar;

(ii) G contains neither K^4 nor $K_{2,3}$ as a minor;

(iii) G contains neither K^4 nor $K_{2,3}$ as a topological minor.

Hint. You may assume Kuratowski's Theorem is true.

Bonus problem. Suppose that G is maximally planar and has at least six vertices. Prove that for any non-adjacent vertices u, v, the graph $G + uv$ contains *both* a TK^5 and a $TK_{3,3}$.