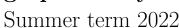
## Advanced and algorithmic graph theory





## Exercise sheet 5

Exercises for the exercise session on 13/06/2022

**Problem 5.1.** Prove that the recursive-largest-first algorithm colours all bipartite graphs optimally and show that it can be implemented to run in time O(nm).

**Problem 5.2.** Given a non-empty graph G, denote by  $P_G: \mathbb{N} \to \mathbb{N}$  the function that maps each  $k \in \mathbb{N}$  to the number of k-colourings of G (recall that we assume the set of colours of a k-colouring to be  $\{1, \ldots, k\}$ ).

(a) Use induction on ||G|| to prove that  $P_G$  is a polynomial of the form

$$P_G(k) = k^{|G|} - ||G|| k^{|G|-1} + \sum_{i=1}^{|G|-2} a_i k^i.$$

 $(P_G \text{ is also called the chromatic polynomial of } G.)$ 

(b) Describe how to determine the chromatic polynomial of a graph algorithmically. What running time do you need?

**Problem 5.3.** Prove directly (that is, without using any results about edge-colourings from the lecture) that every k-regular bipartite graph is k-edge-colourable. Prove that this implies Theorem 4.22, i.e.  $\chi'(G) = \Delta(G)$  for every bipartite graph.

**Problem 5.4.** Describe an algorithm that finds, for every input graph G, an edge-colouring of G with at most  $\chi'(G) + 1$  colours. What running time can you achieve? Hint. Vizing's theorem and its proof.

**Problem 5.5.** For every  $k \in \mathbb{N}$ , construct a bipartite graph  $G_k$  and an assignment of lists that shows that  $G_k$  is *not* k-choosable.

**Problem 5.6.** A total colouring of G is a function  $c: V(G) \cup E(G) \to S$  such that  $c|_{V(G)}$  and  $c|_{E(G)}$  are vertex- and edge-colourings, respectively, and in addition no edge has the same colour as one of its end vertices. We write  $\chi_t(G)$  for the least k for which there exists a total colouring of G with k colours.

Prove that the list colouring conjecture would imply  $\Delta(G) + 1 \leq \chi_t(G) \leq \Delta(G) + 3$ . (The total colouring conjecture asserts that even  $\chi_t(G) \leq \Delta(G) + 2$ .)