Analytic combinatorics
Summer term 2022

Exercise sheet 1

Exercises for the exercise session on 10 March 2022

Problem 1.1. Consider a sequence $x = (x_0 = 0, x_1, \ldots, x_{2n-1}, x_{2n} = 0)$ of non-negative integers satisfying $|x_i - x_{i-1}| = 1$ for $1 \leq i \leq 2n$. This represents an excursion that takes place in the upper half-plane, also known as *Dyck paths of length* $2n$. Let $\mathcal{D}$ be the class of Dyck paths and let $D(z)$ be its ordinary generating function.

(a) Express $\mathcal{D}$ in terms of $\mathcal{D}$ and basic constructions (e.g. combinatorial sum etc.) and derive the corresponding recursive formula for $D(z)$ (i.e. express $D(z)$ in terms of $D(z)$ and z).

(b) Solve the recursive formula to derive a closed expression for $D(z)$.

(c) Derive a closed formula for $[z^{2n}]D(z)$.

Problem 1.2. A *bridge* is a word over $\{-1, +1\}$ whose values of its letters sum to 0. Note that a bridge represents a walk that wanders above and below the horizontal line, but its final altitude is constrained to be 0. Let $\mathcal{B}$ the class of bridges and let $B(z)$ be its ordinary generating function.

(a) Express $\mathcal{B}$ in terms of the class $\mathcal{D}$ of Dyck paths and basic constructions.

(b) Derive a closed expression for $B(z)$ and determine $[z^n]B(z)$.

Problem 1.3. Consider the number of ways a string of $n \geq 1$ identical letters, say x, can be 'bracketed'. The rule is best stated recursively: x itself is a bracketing and if $\sigma_1, \ldots, \sigma_k$ with $k \geq 2$ are bracketed expressions, then the k-ary product $(\sigma_1 \cdots \sigma_k)$ is a bracketing. For instance $(((xx)x(xxx))((xx)(xx)x))$ is a bracketing of 11 letters. Let $\mathcal{S}$ denote the class of all bracketings, where size is taken to be the number of instances of x, and let $S(z)$ denote the ordinary generating function of $\mathcal{S}$.

(a) Express $\mathcal{S}$ in terms of $\mathcal{S}$ and basic constructions (e.g. combinatorial sum etc.).

(b) Derive the corresponding recursive formula for $S(z)$.

(c) Solve the recursive formula to derive a closed expression for $S(z)$.

(d) Can the "basic" methods to determine coefficients we know so far – the generalised binomial theorem and Lagrange inversion – be applied to determine an explicit formula for $[z^n]S(z)$? If so, what formula do we get? If they are not applicable, why not?

Problem 1.4. For a fixed integer $r \geq 2$, let $\mathcal{R}$ be the class of r-nary trees, that is, (unlabelled) plane rooted trees in which every vertex either has precisely r children or none at all. Denote by $R(z)$ the ordinary generating function of $\mathcal{R}$.

(a) Argue directly (i.e. without using generating functions) that the number n of vertices in any r-nary tree always satisfies $n \equiv 1 \bmod r$.

(b) Express $\mathcal{R}$ in terms of $\mathcal{R}$ and basic constructions and derive the corresponding recursive formula for $R(z)$.

(c) Use Lagrange inversion to determine a closed expression for $[z^{kr+1}]R(z)$.

(d) Apply Stirling's formula to derive an asymptotic formula for $[z^{kr+1}]R(z)$.

Problem 1.5. Let $\mathcal{U}$ be the class of unary-binary trees, that is, (unlabelled) plane rooted trees in which every vertex has 0, 1 or 2 children. Denote by $U(z)$ the ordinary generating function of $\mathcal{U}$.

(a) Express $\mathcal{U}$ in terms of $\mathcal{U}$ and basic constructions and derive the corresponding recursive formula for $U(z)$.

(b) Use Lagrange inversion to determine a sum formula for $[z^n]U(z)$.

(c) In order to deduce a (rough) estimation for the asymptotic behaviour of $[z^n]U(z)$, figure out which summand in the sum formula is the largest.