Exercise sheet 2

Exercises for the exercise session on 24 March 2022

The following is a generalisation of Lagrange inversion.

Theorem 1. Let $A(z), \phi(u)$ be formal power series that satisfy the conditions of Lagrange inversion such that $A(z) = z\phi(A(z))$. For any formal power series $\psi(u)$,

$$[z^n]\psi(A(z)) = \frac{1}{n}[u^{n-1}]\left(\psi'(u)(\phi(u))^n\right).$$

Problem 2.1. Denote by $\mathcal{F}_n$ and $\mathcal{H}_n$ the classes of all mappings $[n] \to [n]$ and of all such mappings without fixed points, respectively. We know from the lecture that the exponential generating functions of $\mathcal{F} := \bigcup_n \mathcal{F}_n, \mathcal{H} := \bigcup_n \mathcal{H}_n$ satisfy

$$F(z) = \frac{1}{1 - C(z)} \qquad \text{and} \qquad H(z) = \frac{e^{-C(z)}}{1 - C(z)},$$

respectively, where $C(z)$ is the exponential generating function of Cayley trees.

(a) Determine $f_n := |\mathcal{F}_n|$ and $h_n := |\mathcal{H}_n|$ by directly counting mappings.

(b) Use Theorem 1 to derive a sum formula for the coefficient $[z^n]F(z)$ and to show the relation $[z^n]H(z) = \frac{n-1}{n}[z^{n-1}]F(z)$.

(c) Verify that (a) and (b) yield the same results.

Problem 2.2. Denote by $\mathcal{U}$ the class of labelled *unicyclic* graphs, that is, connected graphs with precisely one cycle.

(a) Express $\mathcal{U}$ in terms of the class $\mathcal{C}$ of Cayley trees and basic constructions and derive the corresponding expression in terms of exponential generating functions.

(b) Use Theorem 1 to determine (as sum formula for) the number of unicyclic graphs on n vertices.

Problem 2.3. The bivariate exponential generating function of permutations counted according to both the number of elements (marked by z) and the number of cycles (marked by u) is

$$P(z, u) = (1 - z)^{-u} = \sum_{n=0}^{\infty} \binom{u + n - 1}{n} z^n.$$

Denote by X_n the number of cycles in a random permutation, which is chosen uniformly at random from all permutations of size n. We know from the lecture that

$$\mathbb{E}[X_n] = H_n := \sum_{k=1}^{n} \frac{1}{k}.$$

Determine the second factorial moment

$$\mathbb{E}[X_n(X_n - 1)]$$

and deduce from the result that $\mathbb{V}[X_n] \le \mathbb{E}[X_n]$.

Problem 2.4. Let C_n and P_n be a Cayley tree and a plane rooted tree, respectively, chosen uniformly at random from all Cayley trees or plane rooted trees on n vertices, respectively. Determine

(a) the expected degree of the root of C_n;

(b) the expected degree of the root of P_n.