Analytic combinatorics
Summer term 2022

Exercise sheet 3

Exercises for the exercise session on 6 April 2022

All asymptotic expressions on this sheet should be determined up to a multiplicative error of $1 + O(1/n)$. *(If you can provide a stronger error term, even better, but not necessary.)*

Problem 3.1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function and suppose that there is a point $z_0 \in \mathbb{C} \setminus \{0\}$ and a real number $R > |z_0|$ such that

- $f(z_0) = 0$;
- $f(z) \neq 0$ for all $z \neq z_0$ with $|z| < R$;
- $f'(z_0) \neq 0$.

(a) Prove that there exists a function $H(z)$ that is holomorphic on the open disc of radius R around the origin and satisfies

$$\frac{1}{f(z)} = \frac{1}{f'(z_0)(z - z_0)} + H(z).$$

(This in particular proves the missing part in the example from the lecture about counting surjections.)

(b) Derive an asymptotic expression for $[z^n]\frac{1}{f(z)}$.

Problem 3.2. For $r, n \in \mathbb{N}$ and a surjection $\phi\colon [n] \to [r]$, we say that ϕ is *even* if all preimages $\phi^{-1}(i)$ for $i \in [r]$ have even size. Analogously, we call ϕ *odd* if each $\phi^{-1}(i)$ has odd size. Denote by $\mathcal{E}_n^{(r)}$ and $\mathcal{O}_n^{(r)}$ the combinatorial classes of all even and of all odd surjections, respectively. Consider the exponential generating functions $E(z)$ and $O(z)$ of

$$\mathcal{E} = \sum_{n\geq 1}\sum_{r\geq 0} \mathcal{E}_n^{(r)} \qquad \text{and} \qquad \mathcal{O} = \sum_{n\geq 1}\sum_{r\geq 0} \mathcal{O}_n^{(r)},$$

respectively.

(a) Derive closed expressions for $E(z)$ and $O(z)$

(b) Determine asymptotic expressions for $[z^n]E(z)$ and $[z^n]O(z)$.

Problem 3.3. Let $B(z) = \sum_n B_n z^n$ and $S(z) = \sum_n S_n z^n$ denote the ordinary generating functions for the classes of binary strings with no consecutive 0's (note: the empty string is included in this class) and the class of bracketings, respectively. Recall from Problem 1.3 that

$$S(z) = \frac{1 + z - \sqrt{z^2 - 6z + 1}}{4}.$$

Derive a closed expression for $B(z)$ and use the transfer theorem to find asymptotic expressions for B_n and S_n.

Problem 3.4. An *alignment* is a sequence of cycles (of labelled objects). Denote by $A(z) = \sum_n A_n \frac{z^n}{n!}$ the exponential generating function for the class of alignments. Moreover, recall from the lecture that the ordinary generating function $T(z) = \sum_n T_n z^n$ of triangulations is given by

$$T(z) = \frac{1 - \sqrt{1 - 4z}}{2z} .$$

Derive

(a) a closed expression for $A(z)$;

(b) an asymptotic expression for A_n;

(c) an asymptotic expression for T_n, using the transfer theorem. Compare the result with the asymptotic expression obtained by applying Stirling's formula to the closed expression $T_n = \frac{1}{n+1}\binom{2n}{n}$.

Problem 3.5. Let G_n denote the number of vertex-labelled 2-regular (simple) graphs on vertex set $[n]$, all whose components have even size.

(a) Derive a closed expression for the exponential generating function $G(z) = \sum_n \frac{G_n}{n!} z^n$.

(b) Use the transfer theorem for multiple singularities to derive an asymptotic expression for G_n.