## Exercise sheet 4

Exercises for the exercise session on 5 May 2022

**Problem 4.1.** For $r \geq 2$, let $\mathcal{C}_r$ be the class of Cayley trees in which every vertex has at most $r$ children. Denote by $C_r(z)$ the exponential generating function of $\mathcal{C}_r$.

(a) Show that the singular inversion theorem can be applied to $C_r(z)$ for every $r$ and determine an asymptotic formula for $[z^n]C_r(z)$ in the special case $r = 2$.

(b) Prove that the dominant singularity $\rho_r$ of $C_r(z)$ converges to $\frac{1}{e}$ for $r \to \infty$.

**Problem 4.2.** Let $A(z)$ be a generating function of a combinatorial class $\mathcal{A}$ and let $\phi(u) = \sum_k \phi_k u^k$ be a power series such that $A(z) = z\phi(A(z))$. Set $G(z, w) := z\phi(w)$.

(a) Verify that $A, \phi$ satisfy the conditions of the singular inversion theorem if and only if $A, G$ satisfy the conditions of the implicit function scheme.

(b) Compare the singular expansions of $A(z)$ and (given the additional assumption of aperiodicity) the asymptotic formula for $[z^n]A(z)$ provided by the two theorems.

**Problem 4.3.** The ordinary generating function $T(z)$ of the class of triangulations of convex polygons satisfies

$$T(z) = G(z, T(z)) := 1 + zT(z)^2.$$

Check the conditions of the implicit function scheme for the functions $T(z)$ and $G(z, w)$. Which are satisfied and which are not? Define an auxilary function $\tilde{T}(z)$ to which the implicit function scheme applies and use this to derive asymptotic formulae for $[z^n]\tilde{T}(z)$ and for $[z^n]T(z)$.

**Problem 4.4.** A *rooted dissection* of a convex polygon with a distinguished edge (the *root*) is a set of non-crossing diagonals of the polygon. Let $\mathcal{D}_n$ be the class of rooted dissections of regular $(n+2)$-gons. The ordinary generating function $D(z)$ of $\mathcal{D} = \bigcup_n \mathcal{D}_n$ satisfies

$$D(z) = (1 + D(z)) \left( \frac{1}{1 - z(1 + D(z))} - 1 \right).$$

Use the implicit function scheme to derive an asymptotic formula for $[z^n]D(z)$.

**Problem 4.5.** Denote by $T(z)$ the exponential generating function of the class $\mathcal{T}$ of labelled (unrooted) trees in which all vertices have degree 1 or 3. (Observe that every such tree has even order.)

(a) The dissymmetry theorem holds for $\mathcal{T}$. (Convince yourself that this is true.) Apply the dissymmetry theorem to derive an expression of $T(z)$ in terms of $T^{\circ-\circ}(z)$.

(b) Determine the dominant singularities of $T^{\circ-\circ}(z)$ and singular expansions near those singularities.

(c) Use (a) and (b) to determine an asymptotic formula for $[z^n]T(z)$ for $n$ even.

*Hint.* For (a) and (b), you will in particular need to express $\mathcal{T}^{\circ}$ and $\mathcal{T}^{\circ-\circ}$, respectively, in terms of $\mathcal{T}^{\circ-\circ}$ and basic constructions. To do this, consider what is left from the tree if you delete the marked vertex or edge. How can we modify the remaining components so as to result in an element of $\mathcal{T}^{\circ-\circ}$?