Analytic combinatorics

Summer term 2022

Exercise sheet 5

Exercises for the exercise session on 18 May 2022

Problem 5.1. Suppose that $G(z), H(z)$ are both Hayman admissible with the same radius of convergence ρ. Prove that the function $F(z) := G(z)H(z)$ satisfies the capture condition of Hayman admissibility and that there exist $\rho_0 \in (0, \rho)$ and functions

$$\theta_0^-, \theta_0^+ : (\rho_0, \rho) \to (0, \pi)$$

such that $F(z)$ satisfies the locality condition for $|\theta| \leq \theta_0^-$ and the decay condition for $|\theta| > \theta_0^+$. (Meta question: Where does the difficulty lie in finding a single function θ_0 that works for both conditions?)

Problem 5.2. The class $\mathcal{P}$ of permutations σ with $\sigma^3 = \mathrm{id}$ has exponential generating function

$$P(z) = \exp\left(z + \frac{z^3}{3}\right).$$

Prove that $P(z)$ is Hayman admissible.
Remark. You can apply the necessary condition $b(s)^{-1/2} \ll \theta_0 \ll c(s)^{-1/3}$ to find a suitable function $\theta_0 = \theta_0(s)$, but afterwards you should check that the locality condition and the decay condition in fact hold for this θ_0.

Problem 5.3. Let $P(z)$ be as in Problem 5.2.

(a) Approximate the unique positive solution s_0 of the saddle-point equation

$$s_0 \frac{P'(s_0)}{P(s_0)} = n$$

in the following way. Substitute $s_0 = \alpha_1 n^{\beta_1} + \alpha_2 n^{\beta_2} \pm n^{\beta_1 - 1 - \varepsilon}$ (with $\beta_1 > \beta_2$ and $\varepsilon > 0$) in the saddle-point equation and choose the constants so that the left-hand side becomes $n \pm cn^{-\varepsilon} + o(n^{-\varepsilon})$ with some constant $c > 0$.

Argue that this implies $s_0 = (1 + o(1))(\alpha_1 n^{\beta_1} + \alpha_2 n^{\beta_2})$.

(b) Use (a) to derive an asymptotic formula for $[z^n]P(z)$.

Problem 5.4. Denote by $P(z)$ the ordinary generating function of plane rooted trees, and let $F(z) := e^{P(z)}$. Use saddle-point estimates of large powers to determine asymptotic formulae for $[z^n]P(z)$ and $[z^n]F(z)$.

Hint: First apply Lagrange inversion to express $[z^n]P(z)$ and $[z^n]F(z)$ by an expression to which saddle-point estimates of large powers can be applied.

Problem 5.5. Denote by $S(z)$ the ordinary generating function of bracketings (see also Problems 1.3 and 3.3). Apply the standard Lagrangean framework to derive an asymptotic expression for $[z^n]S(z)$.