Discrete and algebraic structures
Winter term 2021/22

Exercise sheet 1

Exercises for the exercise session on 14/10/2021

For all problems on this sheet, let $P, U, V, W, W', U_1, \ldots, U_m, V_1, \ldots, V_m$ be finite-dimensional vector space over a field K.

Problem 1.1. Let a multilinear map $\varphi \colon V_1 \times \cdots \times V_m \to W$ and linear maps $T \colon W \to W'$ and $T_i \colon U_i \to V_i$ for each i be given.

(a) Prove that $T \circ \varphi \colon V_1 \times \cdots \times V_m \to W'$ is multilinear.

(b) We define $\psi \colon U_1 \times \cdots \times U_m \to W$ by

$$\psi(u_1, \ldots, u_m) = \varphi(T_1(u_1), \ldots, T_m(u_m)).$$

Show that ψ is multilinear.

Problem 1.2. Let $\varphi \colon V_1 \times \cdots \times V_m \to P$ be a multilinear map and suppose that $\psi \colon V_1 \times \cdots \times V_m \to W$ is a tensor map.

(a) Prove that φ is a tensor map if and only there is a linear map $T \colon P \to W$ with $\psi = T \circ \varphi$.

(b) Suppose that φ is a tensor map and show that the map T from (a) is *unique* if and only if $\langle \mathrm{im}\varphi \rangle = P$ and that it can be chosen to be a bijection if and only if $\dim(P) = \dim(W)$ (but not necessarily $\langle \mathrm{im}\varphi \rangle = P$).

Problem 1.3. Let $u_1, \ldots, u_k \in U$, $v_1, \ldots, v_k \in V$, and $w_1, \ldots, w_k \in W$.

(a) Prove that if $\sum_{i=1}^{k} (u_i \otimes v_i)$ has rank k, then both sets $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ are linearly independent.

(b) Suppose that $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ are linearly independent and $w_i \neq 0$ for all $i = 1, \ldots, k$. Show that

$$\sum_{i=1}^{k} (u_i \otimes v_i \otimes w_i) \in U \otimes V \otimes W$$

has rank k.

Problem 1.4. Suppose that, for each $i = 1, \ldots, m$, there are $v_{i,1}, \ldots, v_{i,k} \in V_i$ such that

$$\sum_{j=1}^{k} v_{1,j} \otimes \cdots \otimes v_{m,j} = 0.$$

Prove that if $v_{1,1}, \ldots, v_{1,k}$ are linearly independent, then for each $j = 1, \ldots, k$, at least one of $v_{2,j}, \ldots, v_{m,j}$ is zero.

Problem 1.5. Suppose that the multilinear map $\varphi \colon V_1 \times \cdots \times V_m \to W$ is surjective. Prove that there exists a subspace U of $V_1 \otimes \cdots \otimes V_m$ such that the quotient space $(V_1 \otimes \cdots \otimes V_m)/U$ is isomorphic to W, and each of its equivalence classes contains a decomposable tensor.