Exercise sheet 2

Exercises for the exercise session on 21/10/2021

For all problems on this sheet, let $V, V_1, \ldots, V_m, W_1, \ldots, W_m$ be finite-dimensional vector spaces over a field K.

Problem 2.1. Let bases $E_1 = \{e_{11}, \ldots, e_{1m}\}$ of $\mathbb{C}^m$ and $E_2 = \{e_{21}, \ldots, e_{2n}\}$ of $\mathbb{C}^n$ be given. Then we know from the lecture that

$$E := \left\{e_\gamma^\otimes = e_{1\gamma(1)} \otimes e_{2\gamma(2)} \mid \gamma = (\gamma(1), \gamma(2)) \in \Gamma = \{1, \ldots, m\} \times \{1, \ldots, n\}\right\}$$

is a basis of $\mathbb{C}^m \otimes \mathbb{C}^n$. For $u = \sum_{\gamma\in\Gamma} a_\gamma e_\gamma^\otimes$ and $v = \sum_{\gamma\in\Gamma} b_\gamma e_\gamma^\otimes$, set

$$(u, v) := \sum_{\gamma\in\Gamma} a_\gamma \overline{b_\gamma}.$$

(a) Show that (u, v) is an inner product on $\mathbb{C}^m \otimes \mathbb{C}^n$ and that E is an orthonormal basis with respect to this inner product.

(b) Prove that if E_1, E_2 are the standard bases of $\mathbb{C}^m$ and $\mathbb{C}^n$, then for matrices $A, B \in \mathbb{C}^m \otimes \mathbb{C}^n = \mathbb{C}_{m\times n}$, the inner product (A, B) equals the trace (i.e. the sum of the diagonal entries) of the matrix B^*A (with $B^* = \overline{B}^T$ as usual).

Problem 2.2. Let linear maps $S_i, T_i \colon V_i \to W_i$, $i = 1, \ldots, m$, be given. Without using Problem 2.3, prove the following statements.

(a) $\otimes_{i=1}^m T_i$ is zero if and only if $T_i = 0$ for some i, and it is bijective if and only if each T_i is bijective; in that case, $(\otimes_{i=1}^m T_i)^{-1} = \otimes_{i=1}^m T_i^{-1}$.

(b) $\otimes_{i=1}^m T_i = \otimes_{i=1}^m S_i \neq 0$ if and only if $T_i = c_i S_i \neq 0$ for $i = 1, \ldots, m$ and $\prod_{i=1}^m c_i = 1$.

Problem 2.3. Prove that $\mathrm{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$ is the tensor product of $\mathrm{Hom}(V_1, W_1), \ldots, \mathrm{Hom}(V_m, W_m)$ along the following lines.

- Verify that $\dim\big(\mathrm{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)\big) = \prod_{i=1}^m \dim\big(\mathrm{Hom}(V_i, W_i)\big)$ (i.e. the dimensions fit).
- Show that the map

$$\varphi\colon \mathrm{Hom}(V_1, W_1) \times \cdots \times \mathrm{Hom}(V_m, W_m) \to \mathrm{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$$

  defined by $\varphi(T_1, \ldots, T_m) := \otimes_{i=1}^m T_i$ is multilinear.
- Prove that $\langle \mathrm{im}\varphi\rangle = \mathrm{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$.

Hint. If bases of vector spaces X, Y are given, what is the canonical basis of $\mathrm{Hom}(X, Y)$?

Problem 2.4. For $v \in V$ and $f \in V^*$, define maps $\varphi(v, f), \psi(v, f)\colon V \to V$ by

$$\big(\varphi(v, f)\big)(w) := f(v)w \qquad \text{and} \qquad \big(\psi(v, f)\big)(w) := f(w)v.$$

Check that $\varphi(v, f)$ and $\psi(v, f)$ are linear. Furthermore, prove that both maps $\varphi, \psi\colon V \times V^* \to \operatorname{Hom}(V, V)$ are multilinear. Are they even tensor maps?

Problem 2.5. Let $U := \{v \otimes w - w \otimes v \mid v, w \in V\} \subset V \otimes V$ and define

$$\operatorname{Sym}^2(V) := (V \otimes V)/U.$$

Denote by π the quotient map $V \otimes V \to \operatorname{Sym}^2(V)$. Prove that if $\{e_1, \ldots, e_n\}$ is a basis of V, then $\{\pi(e_i, e_j) \mid 1 \leq i \leq j \leq n\}$ is a basis of $\operatorname{Sym}^2(V)$.