Discrete and algebraic structures
Winter term 2021/22

Exercise sheet 3
Exercises for the exercise session on 04/11/2021

Problem 3.1. Let U, V, W be finite-dimensional vector spaces over a field K.

(a) Prove that $\varphi((u, v) \otimes w) = ((u \otimes w), (v \otimes w))$ is a well-defined map from the set of decomposable tensors in $(U \times V) \otimes W$ to $(U \otimes W) \times (V \otimes W)$ that can be extended to a bijective linear map

$$\varphi \colon (U \times V) \otimes W \to (U \otimes W) \times (V \otimes W).$$

(b) Prove that $v_1, \dots, v_r \in V$ are linearly independent if and only if $v_1 \wedge \cdots \wedge v_r \neq 0$ (as an element of $\bigwedge^r V$).

Problem 3.2. Let A be an abelian group of order m. Then for $k, n \in \mathbb{Z}$ with $k \equiv n \bmod m$, we have $kx = nx$ for all $x \in A$ (why?). Deduce that A is a module over $\mathbb{Z}/m\mathbb{Z}$, where the action $\mathbb{Z}/m\mathbb{Z} \times A \to A$ is given by $(n + m\mathbb{Z}, x) \mapsto nx$. Conclude that every finite abelian group whose order is a prime p can be regarded as a vector space over a field of p elements.

Problem 3.3. For a ring A with unit, we define the *centre* of A as

$$Z(A) := \{x \in A \mid \forall y \in A \colon xy = yx\}.$$

Prove that $Z(A)$ is a ring with unit. For a commutative ring R with unit, prove that A is a (unitary) associative R-algebra if and only if there exists a ring morphism $\varphi \colon R \to Z(A)$ with $\varphi(1_R) = 1_{Z(A)}$.

Problem 3.4. Let M be a left module over a ring R. For non-empty $S \subset M$, we define the *annihilator of S in R* by

$$\mathrm{Ann}_R S = \{r \in R \mid \forall s \in S \colon rs = 0_M\}.$$

(a) Prove that $\mathrm{Ann}_R S$ is a left ideal of R and that it is a two-sided ideal whenever S is a submodule of M.

(b) Suppose that $r, s \in R$ with $r - s \in \mathrm{Ann}_R M$. Prove that $rx = sx$ for each $x \in M$. Deduce that M is also a module over $R/\mathrm{Ann}_R M$ and that the annihilator of M in this ring is $\{0\}$.

Problem 3.5. Let M, N be R-modules and let $f \colon M \to N$ be an R-morphism. Prove that if A is a submodule of M and B is a submodule of N, then

$$f(A \cap f^{-1}(B)) = f(A) \cap B \qquad \text{and} \qquad f^{-1}(B + f(A)) = f^{-1}(B) + A.$$