

Exercise sheet 4

Exercises for the exercise session on 11/11/2021

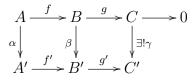
Problem 4.1. Let R be a ring with unit and let M be an R-module. Prove that for any positive integer n, a map $f \colon R^n \to M$ is an R-morphism if and only if there exist $m_1, \ldots, m_n \in M$ such that

$$f((r_1,\ldots,r_n))=r_1m_1+\cdots+r_nm_m.$$

Problem 4.2. Suppose that in the commutative diagram

$$\begin{array}{ccc} A & \stackrel{f}{\longrightarrow} B & \stackrel{g}{\longrightarrow} C & \longrightarrow 0 \\ \alpha & & \beta & \\ \alpha & & \beta & \\ A' & \stackrel{f'}{\longrightarrow} B' & \stackrel{g'}{\longrightarrow} C' \end{array}$$

of *R*-morphisms, the upper row is exact, while the lower row is semi-exact. Show that there exists a unique *R*-morphism $\gamma: C \to C'$ for which the completed diagram



is commutative.

Problem 4.3. Let M be an R-module.

- (a) Prove that if M is finitely generated, then so is every quotient module of M, and that if there is a finitely generated submodule N of M such that M/N is finitely generated, then M is also finitely generated.
- (b) If M is finitely generated, is every submodule of M finitely generated as well?

Problem 4.4. Let R be a commutative ring with unit. We call an R-module M cyclic if there is an element $x \in M$ such that $M = \langle x \rangle$.

Suppose that $M = \langle x \rangle$ is cyclic. Prove that $\operatorname{Ann}_R \{x\} = \operatorname{Ann}_R M$ (the annihilator Ann_R was defined in Problem 3.4) and that $M \cong R/\operatorname{Ann}_R M$ (as *R*-modules). Deduce that two cyclic *R*-modules are isomorphic if and only if they have the same annihilator.

Problem 4.5. Consider the \mathbb{Z} -module $M = \mathbb{R}/\mathbb{Z}$. Prove that

$$N := \{ m \in M \mid \exists z \in \mathbb{Z} \setminus \{0\} \colon zm = 0_M \}$$

is a submodule of M and that $M/N \cong \mathbb{R}/\mathbb{Q}$.