Discrete and algebraic structures
Winter term 2021/22

Exercise sheet 4

Exercises for the exercise session on 11/11/2021

Problem 4.1. Let R be a ring with unit and let M be an R-module. Prove that for any positive integer n, a map $f\colon R^n \to M$ is an R-morphism if and only if there exist $m_1, \dots, m_n \in M$ such that

$$f\big((r_1, \dots, r_n)\big) = r_1 m_1 + \cdots + r_n m_m \,.$$

Problem 4.2. Suppose that in the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] & 0 \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

of R-morphisms, the upper row is exact, while the lower row is semi-exact. Show that there exists a unique R-morphism $\gamma\colon C \to C'$ for which the completed diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\exists!\gamma"] & 0 \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

is commutative.

Problem 4.3. Let M be an R-module.

(a) Prove that if M is finitely generated, then so is every quotient module of M, and that if there is a finitely generated submodule N of M such that M/N is finitely generated, then M is also finitely generated.

(b) If M is finitely generated, is every submodule of M finitely generated as well?

Problem 4.4. Let R be a commutative ring with unit. We call an R-module M *cyclic* if there is an element $x \in M$ such that $M = \langle x \rangle$.

Suppose that $M = \langle x \rangle$ is cyclic. Prove that $\mathrm{Ann}_R\{x\} = \mathrm{Ann}_R M$ (the annihilator Ann_R was defined in Problem 3.4) and that $M \cong R/\mathrm{Ann}_R M$ (as R-modules). Deduce that two cyclic R-modules are isomorphic if and only if they have the same annihilator.

Problem 4.5. Consider the $\mathbb{Z}$-module $M = \mathbb{R}/\mathbb{Z}$. Prove that

$$N := \{m \in M \mid \exists z \in \mathbb{Z} \setminus \{0\} \colon zm = 0_M\}$$

is a submodule of M and that $M/N \cong \mathbb{R}/\mathbb{Q}$.