

## Exercise sheet 4

Exercises for the exercise session on 11/11/2021

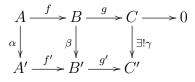
**Problem 4.1.** Let R be a ring with unit and let M be an R-module. Prove that for any positive integer n, a map  $f \colon R^n \to M$  is an R-morphism if and only if there exist  $m_1, \ldots, m_n \in M$  such that

$$f((r_1,\ldots,r_n))=r_1m_1+\cdots+r_nm_m.$$

Problem 4.2. Suppose that in the commutative diagram

$$\begin{array}{ccc} A & \stackrel{f}{\longrightarrow} B & \stackrel{g}{\longrightarrow} C & \longrightarrow 0 \\ \alpha & & \beta & \\ \alpha & & \beta & \\ A' & \stackrel{f'}{\longrightarrow} B' & \stackrel{g'}{\longrightarrow} C' \end{array}$$

of *R*-morphisms, the upper row is exact, while the lower row is semi-exact. Show that there exists a unique *R*-morphism  $\gamma: C \to C'$  for which the completed diagram



is commutative.

**Problem 4.3.** Let M be an R-module.

- (a) Prove that if M is finitely generated, then so is every quotient module of M, and that if there is a finitely generated submodule N of M such that M/N is finitely generated, then M is also finitely generated.
- (b) If M is finitely generated, is every submodule of M finitely generated as well?

**Problem 4.4.** Let R be a commutative ring with unit. We call an R-module M cyclic if there is an element  $x \in M$  such that  $M = \langle x \rangle$ .

Suppose that  $M = \langle x \rangle$  is cyclic. Prove that  $\operatorname{Ann}_R \{x\} = \operatorname{Ann}_R M$  (the annihilator  $\operatorname{Ann}_R$  was defined in Problem 3.4) and that  $M \cong R/\operatorname{Ann}_R M$  (as *R*-modules). Deduce that two cyclic *R*-modules are isomorphic if and only if they have the same annihilator.

**Problem 4.5.** Consider the  $\mathbb{Z}$ -module  $M = \mathbb{R}/\mathbb{Z}$ . Prove that

$$N := \{ m \in M \mid \exists z \in \mathbb{Z} \setminus \{0\} \colon zm = 0_M \}$$

is a submodule of M and that  $M/N \cong \mathbb{R}/\mathbb{Q}$ .