Exercise sheet 5

Exercises for the exercise session on 25/11/2021

Problem 5.1. Let $f, g, h \colon \mathbb{N} \to \mathbb{R}^+$. Prove or disprove the following claims.

(i) $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$;

(ii) $f(n) = o(g(n))$ if and only if $\frac{1}{f(n)} = \omega\left(\frac{1}{g(n)}\right)$;

(iii) $f(n) = O(g(n) + h(n))$ if and only if $f(n) = O(g(n))$ or $f(n) = O(h(n))$;

(iv) If $f(n) = o(g(n))$ and $g(n) = O(h(n))$, then $f(n) = o(h(n))$.

Problem 5.2. Let $f, g, h \colon \mathbb{N} \to \mathbb{R} \setminus \{0\}$ be given such that

$$f(n) = O(h(n)), \quad g(n) = O(h(n)), \quad \text{and} \quad h(n) = o(1).$$

Prove that

$$f(n) + g(n) = O(h(n)), \quad f(n) \cdot g(n) = o(h(n)), \quad \text{and} \quad \frac{1}{1 + f(n)} = 1 + O(h(n)).$$

Problem 5.3. Use Stirling's formula and Problem 5.2 to prove that if α is a constant such that $0 < \alpha < 1$ and $\alpha n \in \mathbb{N}$, then

$$\binom{n}{\alpha n} = \left(1 + O\left(\frac{1}{n}\right)\right) \frac{1}{\sqrt{2\pi\alpha(1-\alpha)n}} \left(\alpha^\alpha (1-\alpha)^{1-\alpha}\right)^{-n}.$$

Show that this result implies the formula

$$\binom{n}{\alpha n} = 2^{H(\alpha)n + O(\log_2 n)}$$

from the lecture, where $H(x) = -x \log_2 x - (1-x)\log_2(1-x)$.

Food for thought for discussion in class (will not influence points for this problem): How big is the mistake we make in the first formula by assuming that αn is an integer? Can we bound this mistake so as to show that the second formula is also true when we round αn to the nearest integer on the left-hand side?

Problem 5.4. Use only the basic operations for formal power series (sum, product, differentiation, integration) and the identity

$$\frac{1}{1-z} = \sum_{n \geq 0} z^n,$$

to determine the complex functions that correspond to the following power series.

(i) $\displaystyle\sum_{n \geq 0} \frac{n}{n+1} z^n$ (ii) $\displaystyle\sum_{n \geq 0} \left(\sum_{k=1}^{n} \frac{1}{k}\right) z^n$ (iii) $\displaystyle\sum_{n \geq 2} n^2 z^n$

Problem 5.5. Let $A(z) = \sum_{n\geq 0} a_n z^n$ be a formal power series.

(a) Prove that $A(z)$ has a reciprocal if and only if $a_0 \neq 0$. Also prove that the reciprocal is unique if it exists.

(b) Suppose that all a_n are non-negative integers and prove that the infinite sum

$$B(z) := 1 + A(z) + A(z)^2 + \cdots$$

used in the sequence contruction of the symbolic method is a well-defined formal power series (i.e. it can be written as $B(z) = \sum_{n\geq 0} b_n z^n$) if and only if $a_0 = 0$. Furthermore, show that $B(z)$ is the reciprocal of $1 - A(z)$.