Discrete and algebraic structures
Winter term 2021/22

Exercise sheet 6
Exercises for the exercise session on 02/12/2021

Problem 6.1. Let $\mathcal{D}$ and $\mathcal{I}$ be the class of *derangements* (permutations without fixed-points) and *involutions* (self-inverse permutations), respectively.

(a) Use the symbolic method to determine the exponential generating function $D(z)$ of $\mathcal{D}$ and derive a sum formula for $[z^n]D(z)$.

(b) Use the symbolic method to determine the exponential generating function $I(z)$ of $\mathcal{I}$ and derive a sum formula for $[z^n]I(z)$.

Problem 6.2. Suppose that the complex function $A(z)$ satisfies

$$z = \frac{A(z)}{1 - A(z)}$$

for all z in some open ball around 0. Determine the coefficients $[z^n]A(z)$ in two different ways: once via Lagrange Inversion and then a second time by deducing the explicit form of $A(z)$ from the above equation.

Problem 6.3. Denote by $T(z)$ the exponential generating function of rooted labelled trees and let $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Determine $[z^n]\exp(\alpha T(z))$ for all integers $n \geq 1$ and deduce from the result that

$$(\alpha + \beta)(n + \alpha + \beta)^{n-1} = \alpha\beta \sum_{k=0}^{n} \binom{n}{k} (k + \alpha)^{k-1}(n - k + \beta)^{n-k-1}$$

for all such α, β, n with $\alpha + \beta \neq 0$.

Problem 6.4. A *forest* is a graph that is the disjoint union of trees. These trees are called *components* of the forest.
Let $\mathcal{F}$ be the class of rooted labelled forests, that is, every component is a rooted labelled tree. Let $\mathcal{G}$ be the class of (unlabelled) plane forests, that is, its components are plane trees and are ordered from left to right. Use Lagrange Inversion and the results known for the classes of rooted labelled trees and plane trees to determine

(a) the number of forests in $\mathcal{F}$, $\mathcal{G}$ with $n \geq 1$ vertices;

(b) for a fixed integer $k \geq 1$, the number of forests in $\mathcal{F}$, $\mathcal{G}$ with $n \geq k$ vertices and precisely k components.

Problem 6.5. Consider an unlabelled tree with a root vertex r. Every vertex $v \neq r$ has a unique neighbour w that is closer to r than v. In this case, we say that v is a *child* of w. (Unlike for binary trees or plane trees, there is no ordering among the children of a fixed vertex.) If $u \neq v$ is also child of w, we call u, v *siblings*.
For an integer $k \geq 1$, let C be a set of k colours und let $\mathcal{T}_k$ be the class of unlabelled rooted trees in which every vertex apart from the root is coloured with a colour in C (i.e. r has no colour). Determine the number of

(a) trees in $\mathcal{T}_k$ with $n \geq 1$ vertices in which there are no two siblings with the same colour;

(b) trees in $\mathcal{T}_k$ with $n \geq m+1$ vertices in which the root has precisely m children (where m is a fixed integer with $1 \leq m \leq k$) and there are no two siblings with the same colour.