Discrete and algebraic structures
Winter term 2021/22

Exercise sheet 7

Exercises for the exercise session on 09/12/2021

Problem 7.1. Let $c \in \mathbb{C}$, $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and an analytic function $f\colon \mathbb{C} \to \mathbb{C}$ be given. Determine the asymptotic value of $[z^{2n}]f(z^2)(1-z^2)^{-\alpha}$

(i) directly, i.e. by applying singularity analysis to $f(z^2)(1-z^2)^{-\alpha}$;

(ii) by applying singularity analysis to the function $f(z)(1-z)^{-\alpha}$ and using that $[z^{2n}]f(z^2)(1-z^2)^{-\alpha} = [z^n]f(z)(1-z)^{-\alpha}$.

Problem 7.2. Let $\mathcal{G}$ be the class of all 2-regular labelled graphs (see also Example 3.3.9 from the lecture) all whose components have odd numbers of vertices. Use the symbolic method to determine the exponential generating function $G(z)$ of $\mathcal{G}$ and apply singularity analysis to derive the asymptotic number of graphs in $\mathcal{G}$ with n vertices up to a multiplicative error $1 + O(1/n)$.

Problem 7.3. Prove Hall's theorem that a bipartite graph with bipartition $\{A, B\}$ has a matching covering A if and only if

$$|N(S)| \geq |S| \quad \text{for every } S \subseteq A.$$

Also give an example that shows that Hall's theorem fails for graphs with infinitely many vertices.

Problem 7.4. Let k, n be positive integers and let X be a set of size kn. Use a result about graphs from the lecture to prove that for any two partitions

$$X = \biguplus_{i=1}^{n} U_i \qquad \text{and} \qquad X = \biguplus_{i=1}^{n} V_i \qquad \text{with } |U_i| = |V_i| = k \text{ for all } i$$

there exists a common set of representatives $Y \subseteq X$ (i.e. $|U_i \cap Y| = |V_i \cap Y| = 1$ for all i). Show that this is not true if we start with three partitions.

Problem 7.5. Let d, n be positive integers. Prove that every connected graph G on n vertices with $\delta(G) \geq d$ contains a path of length

$$k := \min\{2d, n-1\}$$

and, if $d \geq 2$, a cycle of length at least

$$\ell := \min\{d+1, n\}.$$

Show that this is best possible in the sense that for every choice of d, there are infinitely many values of n for which there exists a connected graph G on n vertices with minimum degree at least d such that G neither contains a path of length $k+1$ nor a cycle of length at least $\ell + 1$.

Hint. For the first part, start by considering some longest path P in G. Where are the neighbours of the first and last vertices of P? If P has shorter length than k, try to build from P a cycle with the same vertex set as P (what kind of configuration would enable us to do this, preferably by only changing few edges?), which can then in turn be extended to a path longer than P.