Discrete and algebraic structures
Winter term 2021/22

Exercise sheet 8

Exercises for the exercise session on 16/12/2021

Problem 8.1. Let $G = (V, E)$ be a plane graph.

(a) Suppose that G is connected and that every face of G has at least $g \geq 3$ edges on its boundary. Show that in this case

$$|E| \leq \frac{g}{g-2}(|V| - 2).$$

Deduce from this that neither K_5 nor $K_{3,3}$ is planar.

(b) Let c denote the number of components of G. Find and prove an equation for the number of faces of G in terms of c, $|V|$, and $|E|$.

Note. It is *not* necessary to reprove the Euler formula for connected plane graphs (Theorem 4.2.3 from the lecture).

Problem 8.2. A graph G is called *outerplanar* if it is planar in a way such that all vertices of G lie on the boundary of the outer face. Prove that the following statements are equivalent.

(i) G is outerplanar;

(ii) G contains neither K_4 nor $K_{2,3}$ as a minor;

(iii) G contains neither K_4 nor $K_{2,3}$ as a topological minor.

Problem 8.3. Let $G = (V, E)$ be a graph. Prove that the following statements are equivalent.

(i) G is a tree;

(ii) G is connected and $|E| = |V| - 1$;

(iii) G is maximally acyclic, i.e. G is acyclic, but adding an edge between any to non-adjacent vertices creates a cycle.

Problem 8.4. Let $G = (V, E)$ be a graph.

(a) Suppose that G contains a matching M consisting of m edges. Prove that there exists a set $S \subset V$ such that there are at least

$$\frac{|E| + m}{2}$$

edges between S and $V \setminus S$.

(b) Suppose that $|E| \leq c^2$ for some fixed $c \in \mathbb{N}$. Prove that it is possible to assign a colour to each vertex of G so that at most $2c$ different colours are used in total and for each edge, its end vertices have different colours.

Note/Hint. Both parts can be proved by determining the expectation of a suitable random variable. For (a), choosing S uniformly at random among all subsets of V would suffice if we aimed for $\frac{|E|}{2}$ edges. In order to increase this value, try to ensure that all edges of M are between S and $V \setminus S$. For (b), first consider a random colouring using only c colours.

Problem 8.5. Let H be a graph with k vertices and $m \geq 1$ edges. Suppose that H is *balanced*, that is, for every subgraph $H' \neq \emptyset$ of H, we have

$$\frac{|E(H')|}{|V(H')|} \leq \frac{m}{k} .$$

Use the first and second moment method to prove that

$$\mathbb{P}[H \text{ is a subgraph of } G(n,p)] \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p = o\left(n^{-\frac{k}{m}}\right), \\ 1 & \text{if } p = \omega\left(n^{-\frac{k}{m}}\right). \end{cases}$$

To this end, for each $S \in \binom{[n]}{k}$, denote by X_S the indicator random variable of the event that S forms a graph in $G(n,p)$ that contains H. Then consider $X = \sum_S X_S$. When determining expectations of random variables, keep in mind that in the end, only the *orders* will matter (hence any constant factors can be ignored).