Advanced and algorithmic graph theory

Summer term 2023

Exercise sheet 1

Exercises for the exercise session on 21/03/2023

Problem 1.1. Let G be a connected graph. Prove that the following statements are equivalent for an edge $e \in E(G)$.

(i) $G - e$ is not connected;

(ii) No cycle in G contains e;

(iii) Every spanning tree of G contains e;

(iv) Every spanning tree of G constructed by depth-first search contains e (independently of the order in which we check vertices in the FOR-loops).

Problem 1.2. Let G be a connected weighted graph. For $i = 1, 2$, denote by T_i the graph that is generated by Algorithm i from the lecture. Reminder:

Alg. 1: Start with $T_1 = (V(G), \emptyset)$. Recursively add to T_1 edges of smallest weight that do not create a cycle.

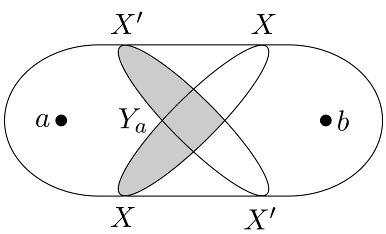
Alg. 2: Start with $T_2 = G$ and recursively delete edges of largest weight that do not disconnect T_2.

Prove that both T_1 and T_2 are spanning trees of G and that both have smallest total weight among all spanning trees of G.

Problem 1.3. Let G be a graph and let a, b be two distinct vertices of G. Suppose that each of the vertex sets $X, X' \subseteq V(G) \setminus \{a, b\}$ is an a-b separator. Denote by C_a and C_b the component of $G - X$ that contains a and b, respectively. Define C'_a and C'_b analogously for X'. Prove that the sets

$$Y_a := (X \cap C'_a) \cup (X \cap X') \cup (X' \cap C_a),$$
$$Y_b := (X \cap C'_b) \cup (X \cap X') \cup (X' \cap C_b)$$

are a-b separators as well.



Problem 1.4. Let G, a, b, X, X', Y_a, Y_b be as in Problem 1.3.

(a) Prove that X is a minimal a-b separator (with respect to containment, i.e. no proper subset of X is an a–b separator) if and only if each vertex in X has neighbours in both C_a and C_b.

(b) Suppose that both X and X' have smallest size among all a–b separators in $V \setminus \{a, b\}$. Prove that Y_a and Y_b are then also smallest a–b separators in $V \setminus \{a, b\}$.

(c) Give an example for which X and X' are minimal a-b separators (w.r.t. containment), but Y_a and Y_b are not minimal.

Problem 1.5. Let G be k-connected, where $k \geq 2$. Prove that G contains a cycle of length at least $\min\{2k, |G|\}$. Show that this statement is best possible in the sense that for *every* $k \geq 2$, there is a k-connected graph with no cycle of length at least $\min\{2k + 1, |G|\}$.

Problem 1.6. Find all mistakes in the following 'proof' of the set version of Menger's theorem. What statement(s) would be necessary to prove in order to complete the proof?

Let S be a smallest A–B separator. We say that a component C of $G - S$ *meets* A if C contains a vertex of A. Denote by G_A the graph that G induces on the union of S and all vertex sets of components of $G - S$ that meet A. Define G_B analogously. By the choice of S and induction, G_A contains $|S|$ disjoint A–S paths, while G_B contains $|S|$ disjoint S–B paths. Joining these paths yields the desired set of $|S|$ disjoint A–B paths.