Advanced and algorithmic graph theory

Summer term 2023

Exercise sheet 2

Exercises for the exercise session on 18/04/2023

(Bonus problems are not counted towards the total number of problems, but solving a bonus problem will earn you a bonus point.)

Problem 2.1. Let G be a 2-connected graph which is not a cycle and let $e \in E(G)$.

(a) Prove that all ear-decompositions of G have the same number k of ears.

(b) Show that there are ear-decompositions $C, P_1, \ldots, P_k$ and $\tilde{C}, \tilde{P}_1, \ldots, \tilde{P}_k$ of G such that e lies on C and on $\tilde{P}_1$.

(c) Prove that e lies on at least $k+1$ distinct cycles in G.

Bonus problem. Is the statement of Problem 2.1(b) best possible? In other words, does there exist, for every choice of integers $k \geq j \geq 2$, a 2-connected graph G and an edge $e \in E(G)$ such that every ear-decomposition of G is of the form $C, P_1, \ldots, P_k$, but no such ear-decomposition satisfies $e \in E(P_j)$?

Problem 2.2. Design an algorithm that constructs ear-decompositions of 2-connected graphs. What running time can you achieve?
Note. Do not write (pseudo-)code for your algorithm, but rather describe in words which steps should be used to find the cycle and the ears of the ear-decomposition.

Problem 2.3. Prove that every graph G with at least two vertices satisfies

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Furthermore, for all integers d, k, l with $1 \leq k \leq l \leq d$, find a graph G with $\kappa(G) = k$, $\lambda(G) = l$, and $\delta(G) = d$.

Problem 2.4. For a graph G, its *line graph* $L(G)$ is defined as the graph on vertex set $E(G)$, in which distinct $e, e' \in E(G)$ are adjacent (as vertices) in $L(G)$ if and only if they are adjacent (as edges) in G.
Use $L(G)$ to prove the edge version of Menger's theorem: For disjoint sets A, B of vertices in a graph G, the largest number of edge-disjoint A–B paths equals the smallest size of an edge set separating A and B.

Problem 2.5. Let G be a bipartite graph with sides A and B.

(a) Let M_A, M_B be matchings in G. Denote by A' the set of vertices in A that M_A covers; define B' analogously for M_B and B. Prove that G has a matching that covers $A' \cup B'$.

(b) Use (a) to show that G has a matching that covers all vertices of maximum degree $\Delta(G)$. Deduce that every r-regular bipartite graph has r edge-disjoint perfect matchings.

Problem 2.6. Let $A_1, \dots, A_n$ be finite sets and $d_1, \dots, d_n \in \mathbb{N}$. Find a necessary and sufficient condition (similar to the marriage condition in Hall's theorem) for the existence of disjoint subsets $B_i \subseteq A_i$ with $|B_i| = d_i$ for all $i \in \{1, \dots, n\}$.