
Exercise sheet 3

Exercises for the exercise session on 02/05/2023

Problem 3.1. Let G be bipartite with sides A and B and suppose that each vertex v has a preference order \leq_v on its set of neighbours. A stable matching M is called *A-optimal* if each vertex $a \in A$ is matched to its “best” neighbour among all vertices that are possible partners for a in stable matchings. (Formally: If $ab \in M$ and $ab' \in M'$ for some stable matching M' , then $b \leq_a b'$.) We define *B-optimal*, *A-pessimal* (worst possible for A) and *B-pessimal* analogously.

- (a) Prove that a stable matching is *A-optimal* if and only if it is *B-pessimal*.
- (b) Prove that every bipartite G has a unique *A-optimal* stable matching.

Problem 3.2. Show that the stable matching generated by STABLE is *A-optimal* (and thus also *B-pessimal*).

Problem 3.3. Prove that Tutte’s Theorem (Theorem 2.8 from the lecture) implies Hall’s Theorem (Theorem 2.2) and that Theorem 2.10 by Gallai and Edmonds implies König’s Theorem (Theorem 2.1).

Hint. For the first part, start by showing that the case $|A| = |B|$ of Hall’s theorem implies the general case. Then prove that in this case, the marriage condition implies the Tutte condition.

Problem 3.4. An edge cover of a graph G that has no isolated vertex is a subset F of edges such that every vertex is incident to at least one edge in F . How are the minimum size of an edge cover and the maximum size of a matching in G related?

Problem 3.5. Prove Corollary 3.9 from the lecture: Every maximally planar graph G with at least four vertices is 3-connected.

Hint. Suppose, for contradiction, that $\{u, v\}$ is a separator of size two. If $uv \in E(G)$, derive a contradiction to the maximality of G . Otherwise, use the non-planarity of $G + uv$ to obtain a contradiction to the planarity of G .

Problem 3.6. A graph is called *outerplanar* if it is planar and has a drawing in which all vertices lie on the boundary of the outer face.

- (a) Find an upper bound (in terms of $|G|$) for the number of edges in an outerplanar graph G .
- (b) Prove that the following statements are equivalent for a graph G .
 - (i) G is outerplanar;
 - (ii) G contains neither K^4 nor $K_{2,3}$ as a minor;
 - (iii) G contains neither K^4 nor $K_{2,3}$ as a topological minor.

Hint for (b). You may assume Kuratowski’s Theorem is true.