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## Exercise sheet 4

Exercises for the exercise session on 31/05/2023

**Problem 4.1.** Prove that for every closed surface, the set of forbidden topological minors is finite.

*Hint.* Start with the set of forbidden minors. For each forbidden minor  $H$ , find a finite number of graphs so that every graph with an  $MH$  contains a subdivision of (at least) one of them. Taking a look at how we found a  $TK^5$  or a  $TK_{3,3}$  in an  $MK^5$  or  $MK_{3,3}$  in the lecture might help.

**Problem 4.2.** Consider a planarity recognition algorithm along the lines of the proof of Kuratowski's theorem from the lecture. Suppose we have already constructed a cycle  $C$  that decomposes  $G$  into several fragments.

- Describe a way for the algorithm to identify all fragments of  $G$  with respect to  $C$ , as well as their attachment sets. What running time can you achieve for this step?
- What running time do you need in order to determine  $O_G(C)$ ?
- Describe how to check whether  $O_G(C)$  is bipartite. (Running time?)
- Suppose that the cycle  $C$  spans  $G$  and is given to us by an oracle (i.e. only constant running time is spent on finding  $C$ ). What would be the running time of the full algorithm in that case?

**Problem 4.3.** Find all mistakes in the following “proof” of the Four Colour Theorem. (In other words, point out which arguments are valid and which are false.)

Suppose, for contradiction, that the Four Colour Theorem is false. Let  $v$  be a vertex of degree  $d := \delta(G) \leq 5$  in a smallest non-4-colourable planar graph  $G$ . Fix a drawing of  $G$  and a 4-colouring  $c$  of  $G - v$ . Denote the neighbours of  $v$  by  $x_1, \dots, x_d$  in the order they lie around  $v$  in the drawing. Furthermore, set  $G_{i,j} := G[c^{-1}(i) \cup c^{-1}(j)]$ . Since  $G$  is not 4-colourable, we know that

$$\text{no 4-colouring of } G - v \text{ uses less than four colours for } N(v). \quad (1)$$

In particular,  $d \geq 4$ . Without loss of generality, we may assume that  $c(x_i) = i$  for  $i = 1, 2, 3, 4$  and, if  $d = 5$ , then  $c(x_5) \in \{1, 2\}$ .

Suppose first that  $d = 4$ . If there is no  $x_1$ - $x_3$  path in  $G_{1,3}$ , then we can recolour  $x_1$  with colour 3 by exchanging the colours in the component of  $G_{1,3}$  that contains  $x_1$  and obtain a colouring  $c'$  that contradicts (1). Otherwise,  $G_{2,4}$  contains no  $x_2$ - $x_4$  path and thus we can recolour  $x_2$  with colour 4, contradicting (1).

Now suppose that  $d = 5$  and  $c(x_5) = 1$ . If there is no  $x_3 - \{x_1, x_5\}$  path in  $G_{1,3}$ , we can recolour  $x_3$  with colour 1. Otherwise, we can recolour  $x_2$  with colour 4 as in the case  $d = 4$ . Either way, we construct a colouring of  $G - v$  that contradicts (1).

Finally, suppose that  $d = 5$  and  $c(x_5) = 2$ . If there is no  $x_1 - x_3$  path in  $G_{1,3}$  or no  $x_1 - x_4$  path in  $G_{1,4}$ , then we can recolour  $x_1$  with colour 3 or 4, respectively, and get a contradiction to (1). Otherwise, there is neither an  $x_2 - x_4$  path in  $G_{2,4}$  nor an  $x_5 - x_3$  path in  $G_{2,3}$ . Thus, we can recolour  $x_2$  with colour 4 and  $x_5$  with colour 3, again a contradiction to (1).

**Problem 4.4.** Let  $G$  be a graph.

- (a) Show that there exists an ordering  $\sigma_0$  of  $V(G)$  such that  $\chi_{\text{Gr}}(G, \sigma_0) = \chi(G)$ .
- (b) Prove that  $\chi_{\text{Gr}}(G, \sigma) \leq \frac{1}{2} + \sqrt{2\|G\| + \frac{1}{4}}$  for every ordering  $\sigma$  of  $V(G)$ .
- (c) Construct, for every positive integer  $n$ , a graph  $G_n$  on  $2n$  vertices and an ordering  $\sigma_1$  of  $V(G_n)$ , for which  $\chi(G_n) = 2$ , but  $\chi_{\text{Gr}}(G_n, \sigma_1) = n$ .

**Problem 4.5.** Prove that the upper bound  $1 + \max_{H \subseteq G} \delta(H)$  for  $\chi(G)$  is strictly larger than  $1 + \frac{1}{2}d(G)$ .

**Problem 4.6.** Along the lines of the proof of Brooks' theorem from the lecture, derive an algorithm that finds, for every connected graph  $G$  that is neither complete nor an odd cycle, a  $\Delta(G)$ -colouring in time  $O(m + n)$ .