Information Theory – SS 2024

Exercise sheet 01 – 21.03.2024

Exercise 1
A coin with possible outcomes 0 (head) and 1 (tail) is thrown 3 times $\left(\Omega = \{0,1\}^3\right)$. What is the minimal σ-algebra containing the following events?

(a) "the outcome of the first throw is 1"
(b) "the sum of the outcomes is even"
(c) "the outcome of the first throw is 1 and the sum of the outcomes is even"
(d) "the outcome of the first throw is 1" and "the sum of the outcomes is even"

Exercise 2
Consider two independent rolls of a fair six sided die. Let i be the outcome of the first roll and j the outcome of the second roll. Consider the following events:

$$A = [i \text{ is even}], \quad B = [i \text{ divides } 6], \quad C = [i + j = 10].$$

(a) Show that $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\ \mathbb{P}(B)\ \mathbb{P}(C)$.
(b) Are the three events independent?

Exercise 3
We are given two urns: urn 1 contains 8 black balls and 2 white ball and urn 2 contains 3 black ball and 9 white balls. First we draw randomly 2 balls from urn 1 and put them in urn 2. Then we draw randomly a ball from urn 2.

(a) Compute the probability that the drawn balls from urn 1 are both black.
(b) Compute the probability that the drawn balls from urn 1 are both black when given that the ball drawn from urn 2 is black.

Exercise 4
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $A_1, A_2, \ldots, A_n \in \mathcal{A}$ events. Show that

(a)

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = k}} \mathbb{P}\left(\bigcap_{i \in I} A_i\right) \right).$$

(b) If the A_i are disjoint and $\bigcup_{i=1}^{n} A_i = \Omega$, then for every $B \in \mathcal{A}$

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \mid A_i)\,\mathbb{P}(A_i).$$

Exercise 5
Let Ω be the set of all finite sequences of zeros and ones.

(a) Find a probability measure $\mathbb{P}$ on $\mathcal{A} = \mathcal{P}(\Omega)$, such that for each finite sequence $\omega \in \Omega$, $\mathbb{P}(\{\omega\}) > 0$.
(b) Is Ω a countable set?

(A *countable* set is a set which can be *enumerated* $\Omega = \{\omega_n \colon n \in \mathbb{N}\}$)

Exercise 6
Let Ω be a sample set and let $(\mathcal{A}_n)_{n \in \mathbb{N}}$ with $\mathcal{A}_n \subseteq \mathcal{P}(\Omega)$ be a family of σ-algebras. Show that:

(a) If $\mathcal{A}_{n+1} \subseteq \mathcal{A}_n$ for every n then $\bigcap_n \mathcal{A}_n$ is a σ-algebra.
(b) If $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ for every n then $\bigcup_n \mathcal{A}_n$ is, in general, not a σ-algebra.