Information Theory – SS 2024

Exercise sheet 02 – 11.04.2024

Exercise 7
The joint distribution of the pair of random variables (X, Y) is given by the following matrix:

(i,j)	1	3	4
0	0.04	0.09	0.27
2	0.06	0.12	0.42

For example, $\mathbb{P}[(X,Y) = (0,1)] = 0.04$.

(a) Calculate the marginal distributions of X and Y.
(b) Are X and Y independent?
(c) Calculate the covariance $\mathrm{Cov}(X,Y)$ of X and Y, which is defined by

$$\mathrm{Cov}(X,Y) = \mathbb{E}\left((X - E(X))(Y - \mathbb{E}(Y))\right).$$

Exercise 8
Let $(X_n)_{n\in\mathbb{N}}$ be an infinite sequence of independent random variables with $\mathbb{P}[X_n = 1] = p_n$ and $\mathbb{P}[X_n = 0] = 1 - p_n$. Show:

(a) $X_n \to 0$ in probability if and only if $p_n \to 0$.
(b) $X_n \to 0$ almost surely if and only if $\sum_{n\geq 1} p_n < \infty$.

(For the second part, you may use the *Borel-Cantelli Lemma* without proof, although it can be shown directly.)

Exercise 9
Show that a random variable $X : \Omega \to \mathbb{R}$ is almost surely constant, if and only if

$$\mathbb{V}(X) = 0\,.$$

(A random variable X is called almost surely constant, if $\mathbb{P}[X = a] = 1$ holds for some $a \in \mathbb{R}$.)

Exercise 10
Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables and let N be an integer-valued random variable independent of the X_i. Let

$$S = X_1 + \cdots + X_N = \sum_{i=1}^{\infty} X_i \mathbf{1}_{i\leq N}.$$

Show that

$$\mathbb{V}(S) = \mathbb{V}(X_1)\,\mathbb{E}(N) + (\mathbb{E}(X_1))^2\,\mathbb{V}(N).$$

Exercise 11
Given a non-negative integer-valued random variable X, show

(a) $\displaystyle\sum_{i=0}^{\infty} i\,\mathbb{P}[X \geq i] = \frac{1}{2}\Big(\mathbb{E}(X^2) + \mathbb{E}(X)\Big),$
(b) $\displaystyle\mathbb{E}(X^2) = \sum_{i=0}^{\infty} i\,(\mathbb{P}[X \geq i] + \mathbb{P}[X > i]).$