Information Theory – SS 2024

Exercise sheet 04 – 25.04.2024

Exercise 16
An urn contains r red, w white, and b black balls. We first draw $k \geq 2$ balls (one by one) from the urn with replacement (putting them back after each draw). Denote their colours by $X_1, \ldots, X_k$. Then we draw k balls from the urn without replacement. Denote their colours by $Y_1, \ldots, Y_k$.

(a) Find $H(X_1, \ldots, X_k)$.

(b) Which is larger, $H(X_1, \ldots, X_k)$ or $H(Y_1, \ldots, Y_k)$? Prove your statement.

(Hint : Clearly $Y_1 \sim X_1$, show that $Y_k \sim X_1$ for all k)

Exercise 17
Suppose you have $n > 2$ coins and among them there may or may not be a counterfeit coin, which is either heavier or lighter than the other coins. You weigh the coins with a balance to determine if there is a heavier or lighter counterfeit coin. Let $k(n)$ be the smallest number of weighings always sufficient to find the counterfeit coin (if any) among the n coins and correctly declare it to be heavier or lighter.

(a) Show that $k(n) \geq \log_3(2n+1)$.

(b) (Difficult) Give a coin weighing strategy for $k = 3$ weighings and $n = 12$ coins.

Exercise 18
Show that the entropy of the probability distribution $(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_n)$ cannot be larger than the entropy of the distribution $(p_1, \ldots, \frac{p_i+p_j}{2}, \ldots, \frac{p_i+p_j}{2}, \ldots, p_n)$. When do we have equality?

Exercise 19
Let $X_1, \ldots, X_n, Y$ be random variables. Prove the chain rule for mutual information, that is, show explicitly that

$$I((X_1, \ldots, X_n); Y) = I(X_1; Y) + I(X_2; Y|X_1) + \ldots + I(X_n; Y|X_1, \ldots, X_{n-1})$$

Give a more general chain rule for expressions of the form $I((X_1, \ldots, X_n); (Y_1, \ldots, Y_m))$.