Information Theory – SS 2024

Exercise sheet 05 – 02.05.2024

Exercise 20
Our goal is to identify a random object X which is distributed in $\mathcal{X}$ with some distribution p. A question Q from a set $\mathcal{Q}$ is asked at random according to distribution r. This results in a deterministic answer $A = A(X, Q)$, that is, there is a deterministic function $(x, q) \mapsto A(x, q)$ from $\mathcal{X}$ to $\mathcal{Q}$. Suppose X and Q are independent. Then $I(X; Q, A)$ is the uncertainty in X removed by the question-answer pair (Q, A).

(a) Show that $I(X; Q, A) = H(A \mid Q)$ and interpret this statement.

(b) Now suppose that two i.i.d. questions $Q_1, Q_2 \sim r$ are asked, yielding answers A_1 and A_2. Show that $I(X; Q_1, A_1, Q_2, A_2) \leq 2I(X; Q_1, A_1)$.

Exercise 21
The *interaction information* of three random variables X_1, X_2, X_3 is defined as

$$I(X_1\,; X_2\,; X_3) := I(X_1\,; X_2) - I(X_1\,; X_2 \mid X_3).$$

(a) Show that the interaction information is symmetric, in the sense that $I(X_1\,; X_2\,; X_3) = I(X_{\sigma(1)}\,; X_{\sigma(2)}\,; X_{\sigma(3)})$ holds for any permutation σ of $\{1, 2, 3\}$.

(b) Give an example of random variables X_1, X_2, X_3 such that $I(X_1\,; X_2\,; X_3) < 0$.

Exercise 22
Consider the Markov chain $(X_n)_{n\geq 0}$ on $\mathcal{X} = \{1, \ldots, 7\}$ with transition matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

(a) Draw the transition graph. Is (X_n) irreducible?

(b) Calculate $\mathbb{P}[X_n = i \mid X_0 = 6]$ for $i \in \mathcal{X}$ and $n \in \{1, 2, 3\}$.

(c) Give a stationary distribution for X_n. Is it unique?

(d) Calculate $\lim_{n\to\infty} \mathbb{P}[X_n = i \mid X_0 = 6]$ for all $i \in \mathcal{X}$.

Exercise 23
Give an example to show that in general, a function of a Markov chain is not necessarily again a Markov chain (i.e., find a Markov chain $(X_n)_{n\geq 0}$ with finite spate space $\mathcal{X}$ and a function $f : \mathcal{X} \to \mathcal{Y}$ such that $(Y_n)_{n\geq 0}$ with $Y_n = f(X_n)$ is not a Markov chain).

Exercise 24
Let $(X_n)_{n\geq 0}$ be an irreducible time homogeneous Markov chain with transition matrix P and stationary initial distribution ν. Let $N \in \mathbb{N}$ and consider the stochastic process $(Y_n)_{n=0}^N$ where $Y_n = X_{N-n}$.

(a) Show that $(Y_n)_{n=0}^N$ is a Markov chain.

(b) Determine the transition matrix for the chain.

(c) Show that the reversed chain is stationary and determine its stationary distribution.

Exercise 25

Consider the *Drunkard's walk* Markov chain with state space $\mathcal{X} = \{0, 1, \ldots, N\}$ and transition matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \beta & 0 & \alpha & 0 & \cdots & 0 & 0 \\ 0 & \beta & 0 & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta & 0 & \alpha & 0 \\ 0 & 0 & \cdots & 0 & \beta & 0 & \alpha \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $0 < \alpha < 1$ is the probability of moving one step from position k to position $k + 1$, and $\beta = 1 - \alpha$ is the probability to move from position k to position $k - 1$, for $k = 1, \ldots, N - 1$.

(a) Given an initial distribution $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the j-th entry, let p_j, for $j = 0, \ldots, N$, be the probability that $X_n = N$ for some $n \geq 0$ (the drunkard reaches home). Find a set of linear equations for the p_j. [Hint: Express p_j in terms of p_{j-1} and p_{j+1}]

(b) Compute p_j for the concrete case where $N = 3$, $j = 1$ and $\alpha = \frac{1}{2}$.