# Information Theory – SS 2024

## Exercise sheet 06 – 16.05.2024

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**Exercise 24**

Let $(X_n)_{n\geq 0}$ be an irreducible time homogeneous Markov chain with transition matrix $P$ and stationary initial distribution $\nu$. Let $N \in \mathbb{N}$ and consider the stochastic process $(Y_n)_{n=0}^N$ where $Y_n = X_{N-n}$.

(a) Show that $(Y_n)_{n=0}^N$ is a Markov chain.

(b) Determine the transition matrix for the chain.

(c) Show that the reversed chain is stationary and determine its stationary distribution.

**Exercise 25**

Consider the *Drunkard's walk* Markov chain with state space $\mathcal{X} = \{0, 1, \ldots, N\}$ and transition matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \beta & 0 & \alpha & 0 & \cdots & 0 & 0 \\ 0 & \beta & 0 & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta & 0 & \alpha & 0 \\ 0 & 0 & \cdots & 0 & \beta & 0 & \alpha \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $0 < \alpha < 1$ is the probability of moving one step from position $k$ to position $k+1$, and $\beta = 1 - \alpha$ is the probability to move from position $k$ to position $k-1$, for $k = 1, \ldots, N-1$.

(a) Given an initial distribution $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the $j$-th entry, let $p_j$, for $j = 0, \ldots, N$, be the probability that $X_n = N$ for some $n \geq 0$ (the drunkard reaches home). Find a set of linear equations for the $p_j$. [Hint: Express $p_j$ in terms of $p_{j-1}$ and $p_{j+1}$]

(b) Compute $p_j$ for the concrete case where $N = 3$, $j = 1$ and $\alpha = \frac{1}{2}$.

**Exercise 26**

Consider the *Weather in Oz* Markov chain $(X_n)_{n\geq 0}$ with state space $\mathcal{X} = \{c, r, s\}$ (clear, rainy, snowy) and transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}.$$

(a) Compute a stationary distribution for the Markov chain.

(b) Compute the entropy rate of the stochastic process.

**Exercise 27**

Consider the experiment of consecutively rolling a fair die. For $n \geq 1$, let $Z_n$ be the RV representing the number of rolls until '6' appears $n$ times.

(a) Is $(Z_n)_{n\geq 1}$ a Markov Chain?

(b) Is $(Z_n)_{n\geq 1}$ stationary?

(c) Compute the entropy rate of $(Z_n)_{n\geq 1}$.

**Exercise 28**

(a) Prove that for every Markov chain $(X_n)_{n\in\mathbb{N}}$ the quantity $H(X_0|X_n)$ is non-decreasing in $n \geq 0$.

(b) Show that for any stationary discrete stochastic process $(X_n)_{n\in\mathbb{Z}}$,

$$H(X_0|X_{-1}, \ldots, X_{-n}) = H(X_0|X_1, \ldots, X_n).$$

**Exercise 29**
Let $(X_n)_{n\geq 1}$ be a stochastic process of i.i.d. Bernoulli RVs with distribution $\mathbb{P}[X_n = 1] = p$ and $\mathbb{P}[X_n = 0] = 1 - p$, for $n \geq 1$. Let $(Y_n)_{n\geq 1}$ be a stationary stochastic process with values in $\{0, 1\}$, such that for $n \geq 3$,

$$Y_n = X_n \oplus Y_{n-1} \oplus Y_{n-2},$$

where $\oplus$ is addition modulo 2. What is the entropy rate of the stochastic process $(Y_n)_{n\geq 1}$?