Information Theory – SS 2024

Exercise sheet 07 – 23.05.2024

Exercise 28

(a) Prove that for every Markov chain $(X_n)_{n\in\mathbb{N}}$ the quantity $H(X_0|X_n)$ is non-decreasing in $n \geq 0$.

(b) Show that for any stationary discrete stochastic process $(X_n)_{n\in\mathbb{Z}}$,

$$H(X_0|X_{-1}, \ldots, X_{-n}) = H(X_0|X_1, \ldots, X_n).$$

Exercise 29
Let $(X_n)_{n\geq 1}$ be a stochastic process of i.i.d. Bernoulli RVs with distribution $\mathbb{P}[X_n = 1] = p$ and $\mathbb{P}[X_n = 0] = 1 - p$, for $n \geq 1$. Let $(Y_n)_{n\geq 1}$ be a stationary stochastic process with values in $\{0, 1\}$, such that for $n \geq 3$,

$$Y_n = X_n \oplus Y_{n-1} \oplus Y_{n-2},$$

where $\oplus$ is addition modulo 2. What is the entropy rate of the stochastic process $(Y_n)_{n\geq 1}$?

Exercise 30
Let $(X_i)_{i\geq 1}$ be a stochastic process with finite state space $\mathcal{X}$. Show that if one of the limits

$$\lim_{n\to\infty} \frac{1}{n} H(X_{k+1}, \ldots, X_{k+n})$$
$$\lim_{n\to\infty} \frac{1}{n} H(X_{k+1}, \ldots, X_{k+n} \mid X_1, \ldots, X_k)$$

exists for some $k \in \mathbb{N}_0$, then both limits exist for every $k \in \mathbb{N}_0$ and all of them coincide.

Exercise 31
Let $(X_n)_{n\geq 0}$ be a time-homogeneous Markov chain with state space $\mathcal{X} = \{0, 1\}$ and transition probabilities: $p(0|0) = 0.3$, $p(1|0) = 0.7$, $p(0|1) = 0.2$, $p(1|1) = 0.8$.

(a) Draw the transition graph.

(b) Compute the stationary distribution ν of the Markov chain.

(c) Let t^0 be the first return time to 0 after starting with $X_0 = 0$. Based on the transition graph, compute $\mathbb{E}(t^0 \mid X_0 = 0)$ as the sum of an infinite series and then find $\nu(0)$ according to the Ergodic Theorem for Markov Chains. Compare it with what you got in part (b).

Exercise 32
Consider again the Markov Chain $(X_n)_{n\geq 0}$ from Exercise 31.

(a) Calculate the entropy rate h.

(b) Given that the Markov chain starts at $X_1 = 0$, find whether the sequences $w_1, w_2 \in \mathcal{X}^{20}$ are in the typical set $A_\varepsilon^{(20)}$, for $\varepsilon = 0.05$:

$$w_1 = 01001010010110110010,$$
$$w_2 = 01110011110111011111.$$

Exercise 33
Consider again the Markov Chain $(X_n)_{n\geq 0}$ from Exercise 31 and suppose that $X_1 = 0$.

(a) Find an integer n and a word $w \in \mathcal{X}^n$ such that w lies in the typical set $A_{0.005}^{(n)}$.

(b) Given $\epsilon > 0$, provide an idea on how to construct a word w lying in the typical set $A_\epsilon^{(n)}$ for some integer n.