Information Theory – SS 2024

Exercise sheet 09 – 13.06.2024

Exercise 39

Let $\mathcal{X} = \{a, b, c, d, e, f, g, h\}$.

(a) Give an example of a prefix code $C : \mathcal{X} \to \{0,1\}^*$ with code word lengths

$$\ell(C(a)) = \ell(C(b)) = \ell(C(c)) = 2,\ \ell(C(d)) = 3,$$
$$\ell(C(e)) = \ell(C(f)) = \ell(C(g)) = 5,\ \ell(C(h)) = 6.$$

(b) Show that the Kraft inequality is a strict inequality for this code, that is,

$$\sum_{x\in\mathcal{X}} 2^{-\ell(C(x))} < 1.$$

(c) Find a binary sequence of length 6 which cannot be decoded.

(d) Find a prefix code $C : \mathcal{X} \to \{0,1\}^*$ such that we have equality in the Kraft inequality.

(e) Find a probability distribution for a random variable $X : \Omega \to \mathcal{X}$ such that your code is optimal and compare the expected code length with $H(X)$.

Exercise 40

Consider the random variable $X : \Omega \to \{x_1, \ldots, x_{10}\}$ with probability distribution

$$p = (\,0.25,\ 0.10,\ 0.13,\ 0.15,\ 0.03,\ 0.05,\ 0.04,\ 0.15,\ 0.08,\ 0.02\,).$$

In a D-ary Huffman code at each step we choose D nodes with the lowest probability and construct their parent with probability the sum of the probabilities of the children. Only at the first step we may merge less than D children: given that $|\mathcal{X}| = N$, at the first step we build the parent of $k = 2 + (N-2) \bmod (D-1)$ children. For example, for $N = 19$ and $D = 5$ we get $k = 2 + 17 \bmod 4 = 3$.

(a) Build a binary Huffman code.

(b) Compute the expected code length of your code and compare it to $H(X)$.

(c) Build a ternary Huffman code over the alphabet $\Sigma = \{0, 1, 2\}$.

(d) Compute the expected code length of your code and compare it to

$$H_3(X) = -\sum_{x\in\mathcal{X}} p(x)\log_3(p(x)).$$

Exercise 41

Let X, Y be random variables with values in $\mathcal{X} = \mathcal{Y} = \{0,1\}$ and let $p_X(0) = q$, $p_X(1) = 1-q$. Let $\mathcal{C} = (\mathcal{X}, P, \mathcal{Y})$ be the channel with input X, output Y and transition matrix

$$P = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}.$$

(a) Compute $I(X,Y)$ in terms of p and q.

(b) For which values of p is $I(X;Y)$ maximal?

(c) For which values of p is $I(X;Y)$ minimal?

Exercise 42

Consider the channel with input X and output $Y = XZ$ where X and Z are independent binary random variables taking values 0 and 1. Let $\alpha = \mathbb{P}[Z = 1]$.

(a) Find the capacity of this channel and the distribution on X maximising $I(X;Y)$.

(b) Now suppose the receiver can observe Z as well as Y. What is the capacity?