Information Theory – SS 2024

Exercise sheet 10 – 20.06.2024

Exercise 43
Imagine you are the commander of an army besieged in a fort and can only communicate with your allies by sending carrier pigeons. Assume that each pigeon can carry one letter (8 bits) and one pigeon can be sent every 5 minutes.

(a) Assuming all pigeons reach their destination safely, what is the capacity of this 'channel' in bits/hour?

(b) Now assume that the enemies try to shoot down the pigeons and manage to hit a fraction α of them. Since the pigeons are sent at a constant rate, the receiver knows which pigeons are missing. What is the capacity of this 'channel'?

(c) Now assume that every time a pigeon is shot down, the enemies send out a dummy pigeon carrying a random letter chosen uniformly from the set of 8-bit letters. What is the capacity of this 'channel'?

Exercise 44
Show that the capacity of the channel with probability transition matrix

$$P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

is achieved by a distribution that places zero probability on one of the input symbols. Calculate the capacity of the channel. Give an intuitive reason why that letter is not used.

Exercise 45
Let $\mathcal{C} = (\mathcal{X}, P, \mathcal{Y})$ be a binary channel which is composed of two binary channels in sequence, such that the output of the first channel $\mathcal{C}_1 = (\mathcal{X}, P_1, \mathcal{Z})$ is the input of the second channel $\mathcal{C}_2 = (\mathcal{Z}, P_2, \mathcal{Y})$. Let the transition matrices be

$$P_1 = \begin{bmatrix} 1-p_1 & p_1 \\ p_1 & 1-p_1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1-p_2 & p_2 \\ p_2 & 1-p_2 \end{bmatrix}.$$

(a) Show that $\mathrm{cap}(\mathcal{C}) \leq \min\{\mathrm{cap}(\mathcal{C}_1), \mathrm{cap}(\mathcal{C}_2)\}$.

(b) Compute the transition matrix P and the channel capacity $\mathrm{cap}(\mathcal{C})$.

(c) Assume that $0 \leq p_1 \leq p_2 < 1/2$. When do we have equality in (a)?

Exercise 46
Consider two independent discrete channels $\mathcal{C}_1 = (\mathcal{X}_1, P_1, \mathcal{Y}_1)$ and $\mathcal{C}_2 = (\mathcal{X}_2, P_2, \mathcal{Y}_2)$. We construct a new channel $\mathcal{C} = (\mathcal{X}_1 \times \mathcal{X}_2,\ P,\ \mathcal{Y}_1 \times \mathcal{Y}_2)$ from $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ will be transmitted in parallel: for $i \in \{1, 2\}$ the input $x_i \in \mathcal{X}_i$ is sent to some $y_i \in \mathcal{Y}_i$ according to the transition matrix P_i.
Compute the channel capacity $\mathrm{cap}(\mathcal{C})$ in terms of $\mathrm{cap}(\mathcal{C}_1)$ and $\mathrm{cap}(\mathcal{C}_2)$.

Exercise 47
Let $\mathcal{C} = (\mathcal{X}, P, \mathcal{Y})$ be a discrete memory-less channel with capacity C. Suppose that this channel is immediately followed by an erasure channel, that erases the output of $\mathcal{C}$ with probability α. This yields a new channel $\mathcal{C}' = (\mathcal{X}, P', \mathcal{Y} \cup \{e\})$ with transition probability

$$p'(y \mid x) = (1-\alpha)p(y \mid x) \text{ and } p'(e \mid x) = \alpha \text{ for } x \in \mathcal{X}, y \in \mathcal{Y}.$$

Compute the capacity of $\mathcal{C}'$.