Information Theory – SS 2024

Exercise sheet 11 – 27.06.2023

Exercise 48
Consider a binary symmetric channel $\mathcal{C}$ with input and output $\{0,1\}$ and crossover probability $p(1 \mid 0) = p(0 \mid 1) = 0.1$. To reduce the error probability, we encode 0 as 000 and 1 as 111, send these three bit words through the channel and then decode the received three bit words by majority (for example, 010 will be decoded as 0). With this coding scheme we can consider the combination of encoder, channel and decoder as a new binary symmetric channel $\mathcal{C}'$.

(a) Calculate the crossover probability of $\mathcal{C}'$
(b) Compare the capacity of $\mathcal{C}'$ to the capacity of $\mathcal{C}$.
(c) Show that for any channel $\mathcal{C}$, considering the encoder, channel and decoder as a new channel $\mathcal{C}'$ will not increase the capacity in bits per transmission of the original channel.

Exercise 49
Given a channel $\mathcal{C} = (\mathcal{X}, P, \mathcal{Y})$ suppose we form a new channel $\mathcal{C}'$ by transmitting the same message X across the channel twice, resulting in an output (Y_1, Y_2) where Y_1 and Y_2 are conditionally independent and conditionally identically distributed given X.

(a) Show that
$$I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2).$$
(b) Conclude that $\mathrm{cap}(\mathcal{C}') \leq 2\mathrm{cap}(\mathcal{C})$.

Exercise 50
Consider the channel $\mathcal{C} = (\mathcal{X}, P, \mathcal{Y})$ with input and output $\mathcal{X} = \mathcal{Y} = \{0,1,2,3\}$ and with probability transition matrix
$$P = \begin{bmatrix} 1-q_1 & q_1 & 0 & 0 \\ q_1 & 1-q_1 & 0 & 0 \\ 0 & 0 & 1-q_2 & q_2 \\ 0 & 0 & q_2 & 1-q_2 \end{bmatrix}.$$

(a) Find the capacity for this channel if $q_1 = q_2 = 1/2$
(b) Let $p_X(x)$ be the input distribution and suppose that $p_X(0) + p_X(1) = p$. Show that the mutual information between the input X and the output Y is
$$I(X;Y) = H(p, 1-p) + pI(X; Y \mid X \in \{0,1\}) + (1-p)I(X; Y \mid X \in \{2,3\}).$$

Exercise 51
Consider again the channel $\mathcal{C} = (\mathcal{X}, P, \mathcal{Y})$ from Exercise 50 and let C_1 and C_2 be the capacities of the two binary symmetric channels with error probabilities q_1 and q_2, respectively.

(a) Show that
$$\max_{p_X(\cdot)} I(X;Y) = \max_{p\in[0,1]} (H(p, 1-p) + pC_1 + (1-p)C_2).$$
(b) Find the capacity C of the channel $\mathcal{C}$ in terms of the capacities C_1 and C_2 and no other parameters.

Exercise 52
Consider a channel with binary inputs that has both errors (with probability ϵ) and erasures (with probability α), and so has transition matrix
$$P = \begin{pmatrix} 1-\alpha-\epsilon & \epsilon & \alpha \\ \epsilon & 1-\alpha-\epsilon & \alpha \end{pmatrix}.$$

Find the capacity of this channel.