## Exercise sheet 1

Exercises for the exercise session on 24/03/2025

**Problem 1.1.** Let $G$ be a connected graph. Prove that the following statements are equivalent for an edge $e \in E(G)$.

  (i) $G - e$ is not connected;

 (ii) No cycle in $G$ contains $e$;

(iii) Every spanning tree of $G$ contains $e$;

(iv) Every spanning tree of $G$ constructed by depth-first search contains $e$ (independently of the order in which we check vertices in the FOR-loops).

**Problem 1.2.** Let $G$ be a connected weighted graph. For $i = 1, 2$, denote by $T_i$ the graph that is generated by Algorithm $i$ from the lecture. Reminder:

**Alg. 1:** Start with $T_1 = (V(G), \emptyset)$. Recursively add to $T_1$ edges of smallest weight that do not create a cycle.

**Alg. 2:** Start with $T_2 = G$ and recursively delete edges of largest weight that do not disconnect $T_2$.

Prove that both $T_1$ and $T_2$ are spanning trees of $G$ and that both have smallest total weight among all spanning trees of $G$.

**Problem 1.3.** Let $G$ be a graph and let $a, b$ be two distinct vertices of $G$. Suppose that each of the vertex sets $X, X' \subseteq V(G) \setminus \{a, b\}$ is an $a$-$b$ separator. Denote by $C_a$ and $C_b$ the component of $G - X$ that contains $a$ and $b$, respectively. Define $C'_a$ and $C'_b$ analogously for $X'$. Prove that the sets

$$Y_a := (X \cap C'_a) \cup (X \cap X') \cup (X' \cap C_a),$$
$$Y_b := (X \cap C'_b) \cup (X \cap X') \cup (X' \cap C_b)$$

are $a$-$b$ separators as well.
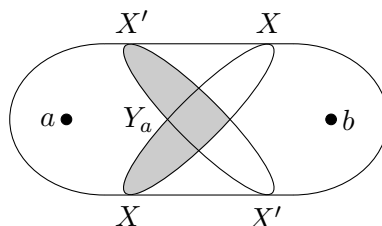
**Problem 1.4.** Let $G, a, b, X, X', Y_a, Y_b$ be as in Problem 1.3.

(a) Prove that $X$ is a minimal $a$-$b$ separator (with respect to containment, i.e. no proper subset of $X$ is an $a$–$b$ separator) if and only if each vertex in $X$ has neighbours in both $C_a$ and $C_b$.

(b) Suppose that both $X$ and $X'$ have smallest size among all $a$–$b$ separators in $V \setminus \{a, b\}$. Prove that $Y_a$ and $Y_b$ are then also smallest $a$–$b$ separators in $V \setminus \{a, b\}$.

(c) Give an example for which $X$ and $X'$ are minimal $a$-$b$ separators (w.r.t. containment), but $Y_a$ and $Y_b$ are not minimal.

**Problem 1.5.** Let $G$ be $k$-connected, where $k \geq 2$. Prove that $G$ contains a cycle of length at least $\min\{2k, |G|\}$. Show that this statement is best possible in the sense that for *every* $k \geq 2$, there is a $k$-connected graph with no cycle of length at least $\min\{2k + 1, |G|\}$.

**Problem 1.6.** Find all mistakes in the following 'proof' of the set version of Menger's theorem. What statement(s) would be necessary to prove in order to complete the proof?

Let $S$ be a smallest $A$–$B$ separator. We say that a component $C$ of $G - S$ *meets* $A$ if $C$ contains a vertex of $A$. Denote by $G_A$ the graph that $G$ induces on the union of $S$ and all vertex sets of components of $G - S$ that meet $A$. Define $G_B$ analogously. By the choice of $S$ and induction, $G_A$ contains $|S|$ disjoint $A$–$S$ paths, while $G_B$ contains $|S|$ disjoint $S$–$B$ paths. Joining these paths yields the desired set of $|S|$ disjoint $A$–$B$ paths.