Advanced and algorithmic graph theory

Summer term 2025

Exercise sheet 4

Exercises for the exercise session on 19/05/2025

Problem 4.1. Prove that for every closed surface, the set of forbidden topological minors is finite.
Hint. Start with the set of forbidden minors. For each forbidden minor H, find a finite number of graphs so that every graph with an MH contains a subdivision of (at least) one of them. Taking a look at how we found a TK^5 or a $TK_{3,3}$ in an MK^5 or $MK_{3,3}$ in the lecture might help.

Problem 4.2. Consider a planarity recognition algorithm along the lines of the proof of Kuratowski's theorem from the lecture. Suppose we have already constructed a cycle C that decomposes G into several fragments.

(a) Describe a way for the algorithm to identify all fragments of G with respect to C, as well as their attachment sets. What running time can you achieve for this step?

(b) What running time do you need in order to determine $O_G(C)$?

(c) Describe how to check whether $O_G(C)$ is bipartite. (Running time?)

(d) Suppose that the cycle C spans G and is given to us by an oracle (i.e. only constant running time is spent on finding C). What would be the running time of the full algorithm in that case?

Problem 4.3. Show that every graph G can be embedded into $\mathbb{R}^3$ with all edges straight.

Problem 4.4. Find all mistakes in the following "proof" of the Four Colour Theorem. (In other words, point out which arguments are valid and which are false.)

Suppose, for contradiction, that the Four Colour Theorem is false. Let v be a vertex of degree $d := \delta(G) \leq 5$ in a smallest non-4-colourable planar graph G. Fix a drawing of G and a 4-colouring c of $G - v$. Denote the neighbours of v by $x_1, \ldots, x_d$ in the order they lie around v in the drawing. Furthermore, set $G_{i,j} := G[c^{-1}(i) \cup c^{-1}(j)]$. Since G is not 4-colourable, we know that

$$\text{no 4-colouring of } G - v \text{ uses less than four colours for } N(v). \tag{1}$$

In particular, $d \geq 4$. Without loss of generality, we may assume that $c(x_i) = i$ for $i = 1, 2, 3, 4$ and, if $d = 5$, then $c(x_5) \in \{1, 2\}$.

Suppose first that $d = 4$. If there is no x_1–x_3 path in $G_{1,3}$, then we can recolour

x_1 with colour 3 by exchanging the colours in the component of $G_{1,3}$ that contains x_1 and obtain a colouring c' that contradicts (1). Otherwise, $G_{2,4}$ contains no x_2–x_4 path and thus we can recolour x_2 with colour 4, contradicting (1).

Now suppose that $d = 5$ and $c(x_5) = 1$. If there is no x_3–$\{x_1, x_5\}$ path in $G_{1,3}$, we can recolour x_3 with colour 1. Otherwise, we can recolour x_2 with colour 4 as in the case $d = 4$. Either way, we construct a colouring of $G - v$ that contradicts (1).

Finally, suppose that $d = 5$ and $c(x_5) = 2$. If there is no x_1–x_3 path in $G_{1,3}$ or no x_1–x_4 path in $G_{1,4}$, then we can recolour x_1 with colour 3 or 4, respectively, and get a contradiction to (1). Otherwise, there is neither an x_2–x_4 path in $G_{2,4}$ nor an x_5–x_3 path in $G_{2,3}$. Thus, we can recolour x_2 with colour 4 and x_5 with colour 3, again a contradiction to (1).

Problem 4.5. Let G be a graph.

(a) Show that there exists an ordering σ_0 of $V(G)$ such that $\chi_{\text{Gr}}(G, \sigma_0) = \chi(G)$.

(b) Prove that $\chi_{\text{Gr}}(G, \sigma) \leq \frac{1}{2} + \sqrt{2\,\|G\| + \frac{1}{4}}$ for *every* ordering σ of $V(G)$.

(c) Construct, for every positive integer n, a graph G_n on $2n$ vertices and an ordering σ_1 of $V(G_n)$, for which $\chi(G_n) = 2$, but $\chi_{\text{Gr}}(G_n, \sigma_1) = n$.

Problem 4.6. Prove that the upper bound $1 + \max_{H \subseteq G} \delta(H)$ for $\chi(G)$ is strictly larger than $1 + \frac{1}{2}d(G)$.