Advanced and algorithmic graph theory

Summer term 2025

Exercise sheet 6

Exercises for the exercise session on 18/06/2025

Problem 6.1. Show that every tournament (an oriented complete graph) contains a directed Hamilton path. Must it contain a Hamilton cycle?

Problem 6.2. Pick one of the proofs of Dirac's theorem and show that it can also be applied to prove Ore's theorem (possibly with some minor changes). Based on the proof, describe an algorithm that finds a Hamilton cycle in a given graph G that satisfies the condition of Ore's theorem. What running time can you achieve? What running time is necessary to check whether G satisfies the condition of Dirac's or Ore's theorem, respectively?

Problem 6.3. Suppose that G is a graph in which $d(x) + d(y) \geq n$ holds for every pair $x \neq y$ of non-adjacent vertices. Prove that the degree sequence of G is hamiltonian.

Problem 6.4. Recall that the square G^2 of a graph G has the same vertex set as G and an edge between v and w iff v and w have distance at most two in G.

(a) Prove that if G is k-connected (for some positive integer k), then G^2 is k-tough.

Hint. Given a set S that separates G^2 into components $C_1, \dots, C_s$, what can we say about the neighbourhoods of $V(C_1), \dots, V(C_s)$ in G?

(b) Find a connected graph G on at least three vertices for which G^2 is not hamiltonian.

Note. Then G^2 is a 1-tough graph that is not hamiltonian.

Problem 6.5. Let T be a tree with $\|T\| \geq 2$. Recall that the *Erdős-Sós Conjecture* states that

$$\mathrm{ex}(n, T) \leq \frac{1}{2}(\|T\| - 1)n.$$

(a) Show that the Erdős-Sós Conjecture is true if $T = K_{1,r}$ (with $r \geq 2$).

(b) Prove the weaker statement

$$\mathrm{ex}(n, T) \leq \|T\|\, n.$$

Hint. If the minimum degree of G is large enough, we might be able to find a copy of T in G one vertex after another.

(c) Show that the Erdős-Sós Conjecture is best possible in the sense that

$$\mathrm{ex}(n, T) \geq \frac{1}{2}(\|T\| - 1)n$$

for infinitely many values of n. (These values might depend on T.)

Problem 6.6. Let r be a positive integer. Show that every $8r^2$-connected graph G contains a TK^r.
Note. A theorem of Mader implies that every graph with average degree at least $16r^2$ *contains a* $8r^2$*-connected subgraph. Thus, this problem implies Theorem 6.7 from the lecture with* $c = 16$.