Exercise sheet 1

Exercises for the exercise session on 30 October 2024

Problem 1. Let $\pi = (\pi(1), \pi(2), \dots, \pi(n))$ be a random permutation of $\{1, 2, \dots, n\}$ generated by sequentially choosing $\pi(i)$ uniformly at random from $\{1, 2, \dots, n\} \setminus \{\pi(1), \dots, \pi(i-1)\}$ for each $i = 1, \dots, n$. Show that π is uniformly distributed over all permutations.
Given a subset $A \subseteq \{1, 2, \dots, n\}$, let $m(A) := \operatorname{argmin}_A\{\pi(i)\}$ be the element of A which has the smallest image under π. Show that $m(A)$ is uniformly distributed over A.

Problem 2. A *k-uniform hypergraph* is a pair $H = (V, E)$ with vertex set V and (hyper)edge set E, where every (hyper)edge is a subset of V containing exactly k elements.
Let $k \geq 2$. Prove that if a k-uniform hypergraph H has fewer than 2^{k-1} edges, then one can colour the vertices of H by two colours so that there is no *monochromatic* edge (that is, an edge whose vertices all have the same colour).
Does the conclusion hold if H has exactly 2^{k-1} edges?

Problem 3. A *tournament* is an orientation of a complete graph, i.e. for every pair of distinct vertices v, w, exactly one of the directed edges (v, w) and (w, v) is present. A *Hamiltonian path* in a tournament is a directed path passing through all vertices. Prove that for every $n \geq 3$, there exists a tournament on n vertices that has more than $n!\, 2^{-n+1}$ Hamiltonian paths.

Problem 4. Prove that for $k, n \in \mathbb{N}$ satisfying

$$\binom{n}{k}\left(1 - \left(\frac{1}{2}\right)^k\right)^{n-k} < 1,$$

there is a tournament on n vertices with the property that for every set A of k vertices, there is some vertex v so that each edge between v and A is directed towards its respective end vertex in A.