Probabilistic method in combinatorics and algorithmics

WS 2024/25

Exercise sheet 10

Exercises for the exercise session on 22th January 2025

Problem 34. Let π be a random permutation of $[n]$, let X be the length of the longest increasing subsequence of π and let m be the median of X.

(a) Show that $\frac{1}{3}\sqrt{n} \leq m \leq 3\sqrt{n}$.

(b) Show that we can generate π by choosing an independent sequence $x_1, \ldots, x_n$ uniformly in $[0,1]$ and letting $\pi(i)$ be the index j of the ith largest x_j, and hence view X as living on a product space.

(c) Show that X is 1-Lipschitz over this product space, and that X is f-certifiable with $f(s) = s$. Deduce that with high probability $X = (1 + o(1))m$.

(You may assume without proof that for any permutation π of $[n]$, the product of the length of the longest increasing sequence and the length of the longest decreasing sequence is at least n.)

Problem 35. Let $\mathcal{P}$ be a decreasing property of graphs and let $0 \leq p_1 \leq p_2 \leq 1$. Show that

$$\mathbb{P}(G(n, p_1) \in \mathcal{P}) \geq \mathbb{P}(G(n, p_2) \in \mathcal{P}).$$

Problem 36. Let $\mathcal{P}$ be a (non-trivial) increasing property of graphs and let p^* be such that $\mathbb{P}(G(n, p^*) \in \mathcal{P}) = \frac{1}{2}$.
Show that

$$\mathbb{P}(G(n, p^*) \in \mathcal{P}) \to \begin{cases} 0 & \text{if } p = o(p^*) \\ 0 & \text{if } p = \omega(p^*). \end{cases}$$

(Hint) : If $p \geq kp^*$ then we can couple k independent copies of $G(n, p^*)$ to be a subgraph of $G(n, p)$. On the otherhand if $kp \leq p^*$ then we can couple k independent copies of $G(n, p)$ to be a subgraph of $G(n, p^*)$.

Problem 37 (Harris' inequality)**.** We say a non-negative real function f of n-vertex graphs is an *increasing graph parameter* if $f(G) \leq f(H)$ whenever $G \subseteq H$.
Suppose X and Y are increasing graph parameters. Show that

$$\mathbb{E}(X(G(n, p)) \cdot Y(G(n, p))) \geq \mathbb{E}(X(G(n, p)))\mathbb{E}(Y(G(n, p))).$$

(Hint: It might be easier to think more generally about functions of $\{0, 1\}^m$ which are 'increasing' and to induct on the dimension m)