# Exercise sheet 2

Exercises for the exercise session on 6 November 2024

**Problem 5.** Let $n \in \mathbb{N}$ and let $\mathcal{F}$ be an inclusion-free family of subsets of $[n] := \{1, 2, \ldots, n\}$ (*inclusion-free* means that no element of $\mathcal{F}$ is a proper subset of another element). Choose a permutation $\sigma$ of $[n]$ uniformly at random and define the random variable

$$X := |\{k \mid \{\sigma(1), \sigma(2), \ldots, \sigma(k)\} \in \mathcal{F}\}|.$$

Determine $\mathbb{E}[X]$, and prove that $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

**Problem 6.** Let $G$ be a bipartite graph with $n$ vertices and suppose that each vertex $v$ has a list $S(v)$ of colours. Prove that if $|S(v)| > \log_2 n$ for each $v$, then we can colour every vertex with a colour from its list so that no two adjacent vertices have the same colour.
*Hint.* Partition the set $\bigcup_v S(v)$ into two random sets.

**Problem 7.** Given two jointly distributed random variables $X$ and $Y$, recall that $\mathbb{E}(X|Y)$ is defined as the random variable such that $\mathbb{E}(X|Y)(\omega) = \mathbb{E}(X|Y = Y(\omega))$ for every $\omega$ in the sample space. Let us define in a similar manner

$$\mathbb{V}(X|Y) = \mathbb{E}((X - \mathbb{E}(X|Y))^2|Y).$$

(a) Prove the *law of total expectation*

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y))$$

(b) Use **??** to prove the *law of total variance*

$$\mathbb{V}(X) = \mathbb{E}(\mathbb{V}(X|Y)) + \mathbb{V}(\mathbb{E}(X|Y)).$$

(c) Suppose $X_1, X_2, \ldots$ is a sequence of i.i.d random variables with the same distribution as $X$ and let $Y$ be an independent random variable taking values in $\mathbb{N}$. Let $Z = \sum_{i=1}^{Y} X_i$ and show that

$$\mathbb{E}(Z) = \mathbb{E}(Y)\mathbb{E}(X) \qquad \text{and} \qquad \mathbb{V}(Z) = \mathbb{E}(Y)\mathbb{V}(X) + \mathbb{V}(Y)\mathbb{E}(X)^2.$$

**Problem 8.** We say a set $\{x_1, x_2, \ldots, x_k\}$ of positive integers has distinct sums if for all subsets $I \subseteq [k]$ the quantities $\displaystyle\sum_{i \in I} x_i$ are distinct. Suppose that $\{x_1, x_2, \ldots, x_k\} \subseteq [n]$ has distinct sums, let $I \subseteq [k]$ be chosen uniformly at random and let $X = \sum_{i \in I} x_i$.
Calculate $\mathbb{E}(X)$ and show that $\mathbb{V}(X) \leq \frac{n^2 k}{2}$. Deduce that with probability at least $\frac{1}{2}$, $X$ lies in some interval of length $O\left(n\sqrt{k}\right)$.
Use the fact that $\{x_1, x_2, \ldots, x_k\}$ has distinct sums to show that $n = \Omega\left(\frac{2^k}{\sqrt{k}}\right)$ and hence $k \leq \log_2 n + \frac{1}{2} \log_2 \log_2 n + O(1)$.