Probabilistic method in combinatorics and algorithmics

WS 2024/25

Exercise sheet 3

Exercises for the exercise session on 13 November 2024

Problem 9. Call an edge in a graph *isolated* if both its end vertices lie in no other edge. Prove that if $\varepsilon \in (0,1)$ is a constant, then

$$\mathbb{P}(G(n,p) \text{ contains an isolated edge}) \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p = (1+\varepsilon)\frac{\ln n}{2n}, \\ 1 & \text{if } p = (1-\varepsilon)\frac{\ln n}{2n}. \end{cases}$$

Problem 10. Suppose an unbiased coin is tossed n times. For $k \le n$, let A_k denote the event that out of these n tosses, there are k consecutive ones with the same outcome (i.e. k consecutive 'heads' or k consecutive 'tails'). Let $\varepsilon > 0$ be constant. Prove that

$$\mathbb{P}(A_k) \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } k \ge (1+\varepsilon)\log_2 n, \\ 1 & \text{if } k \le (1-\varepsilon)\log_2 n. \end{cases}$$

Problem 11. Given $p_1, p_2 \in [0,1]$, let $G(n,p_1)$ and $G(n,p_2)$ be independently generated random graphs. Show that $G(n,p_1) \cup G(n,p_2)$ has the same distribution as $G(n,q)$ for some $q \in [0,1]$ and determine the value of q.
Prove that if $\mathcal{P}$ is an increasing property of graphs, then

$$f\colon [0,1] \to [0,1], \quad f_{\mathcal{P}}(p) := \mathbb{P}[G(n,p) \text{ has property } \mathcal{P}]$$

is an increasing function.

Problem 12. Using the General Lovász Local Lemma (Theorem 4.7), deduce the Symmetric Lovász Local Lemma (Theorem 4.8).