Probabilistic method in combinatorics and algorithmics

WS 2024/25

Exercise sheet 4

Exercises for the exercise session on 20 November 2024

Problem 12. Using the General Lovász Local Lemma (Theorem 4.7), deduce the Symmetric Lovász Local Lemma (Theorem 4.8).

Problem 13. We say that a hypergraph $H = (V, E)$ is *2-colourable* if there exists a colouring of V by two colours so that no edge in E is monochromatic.

(a) Let H be a hypergraph in which every edge has at least k vertices and each edge of H intersects at most $d \geq 1$ other edges. If $e(d+1)2^{1-k} \leq 1$, show that H is 2-colourable.

(b) Suppose that H is k-uniform (every edge has size k) and k-regular (each vertex lies in k edges). If $k \geq 9$, show that H is 2-colourable.

Problem 14. Let $G = (V, E)$ be a bipartite graph with n vertices, and suppose that for each vertex we are given some list $S(v)$ of colours such that for some d

- $|S(v)| \geq 10d$ for each v;
- for each $v \in V$ and $c \in S(v)$ there are at most d neighbours, u, of v with $c \in S(u)$.

Show that we can find a proper colouring of G such that the colour of each vertex v is in $S(v)$.

Problem 15. Let D be a directed graph with minimum outdegree δ and maximum indegree Δ. Show that for any $k \in \mathbb{N}$ with

$$k \leq \frac{\delta}{1 + \log(1 + \delta\Delta)},$$

D contained a directed cycle of length divisible by k.

(Hint : Take a random partition of the vertices into k sets $V_1, \dots, V_k$ and consider the event that a vertex $v \in V_i$ has no outneighbour in V_{i+1}.)