Probabilistic method in combinatorics and algorithmics

WS 2024/25

Exercise sheet 5

Exercises for the exercise session on 27 November 2024

Problem 16. Let G be a d-regular graph with girth at least 5. Show that we can colour the vertices of G with $cd^{\frac{4}{3}}$ colours, for some c sufficiently large, such that all cycles contain edges of at least 3 distinct colours (not insisting the the colouring is proper).

(Hint: Consider the set of events, for each path with 4 edges, that the path contains edges of only 2 colours.)

(*Hard*) Show further that we can find such a colouring which is proper.

Problem 17. Suppose we place m balls in n bins, where each ball chooses its bin uniformly at random and independently from the other balls. Let $\varepsilon > 0$ be a constant.

(a) Prove that if $m = n$,

$$\mathbb{P}\left[\exists \text{ a bin with at least } 1 + \left(\frac{2}{3} + \varepsilon\right) \ln n \text{ balls}\right] = o(1).$$

(b) For $m = n^2$, prove that

$$\mathbb{P}\left[\exists \text{ a bin with at most } n - \sqrt{(2+\varepsilon) n \ln n} \text{ balls}\right] = o(1).$$

(c) If we let ε be a function of n, how fast is it allowed to tend to zero so that your arguments in (a) and (b) still work?

Problem 18. Let $p = \omega\left(\sqrt{\frac{\log(n)}{n}}\right)$. Show that with high probability every edge in $G(n,p)$ is contained in $(1+o(1))p^2 n$ many triangles.

Problem 19. Suppose we place n points uniformly at random in $[0,1]^2$. If we consider $[0,1]^2$ as being composed of $\frac{n}{\log(n)}$ equally sized disjoint squares (ignoring the rounding issues), show that with high probability every square contains a point.
Using the Chernoff bound show that for each point, with high probability, the number of points within distance $10\sqrt{\frac{\log(n)}{n}}$ is at most $c\log(n)$ for an appropriate constant c.
Suppose we form a graph G on these n points by joining each point to the nearest k points to it. Show that, if $k \geq c\log(n)$ then with high probability G is connected.