Exercise sheet 6

Exercises for the exercise session on 4th December 2024

Problem 18. Let $p = \omega\left(\sqrt{\frac{\log(n)}{n}}\right)$. Show that with high probability every edge in $G(n, p)$ is contained in $(1 + o(1))p^2 n$ many triangles.

Problem 19. Suppose we place n points uniformly at random in $[0, 1]^2$. If we consider $[0, 1]^2$ as being composed of $\frac{n}{\log(n)}$ equally sized disjoint squares (ignoring the rounding issues), show that with high probability every square contains a point.
Using the Chernoff bound show that for each point, with high probability, the number of points within distance $10\sqrt{\frac{\log(n)}{n}}$ is at most $c \log(n)$ for an appropriate constant c.
Suppose we form a graph G on these n points by joining each point to the nearest k points to it. Show that, if $k \geq c \log(n)$ then with high probability G is connected.

Problem 20. Let $Z_1, \ldots, Z_m$ be a sequence of mutually independent random variables distributed on $\mathcal{Z}^n$ and let $f \colon \mathcal{Z}^n \to \mathbb{R}$ be a function such that changing the value of one coordinate can change the value of f by at most c. Let $X_i = \mathbb{E}(f(Z_1, \ldots, Z_m) \mid Z_1, \ldots, Z_i)$ be the associated exposure martingale of f. Show that $|X_i - X_{i-1}| \leq c$ for all $1 \leq i \leq m$.
(Hint : Introduce an independent copy of Z_i to express X_i and X_{i-1} as conditional expectations under the same conditioning.)

Problem 21. Suppose we throw n balls into m bins independently and uniformly at random. Let E be the number of empty bins at the end, what is $\mathbb{E}(E)$?
If E_j is the indicator function for the event that the jth bin is empty, then $E = \sum_{j=1}^{m} E_j$. Are the E_j independent?
Show that

$$\mathbb{P}(|E - \mathbb{E}(E)| \geq \lambda\sqrt{n}) \leq 2e^{-\frac{\lambda^2}{2}},$$

and thus conclude that E is tightly concentrated about it's mean if $m = n$.
(Hint: Let Z_i be the bin into which the ith ball is thrown and consider the martingale sequence $X_i = \mathbb{E}(E \mid Z_1, \ldots, Z_i)$.)