Probabilistic method in combinatorics and algorithmics

WS 2024/25

Exercise sheet 7

Exercises for the exercise session on 11th December 2024

Problem 21. Suppose we throw n balls into m bins independently and uniformly at random. Let E be the number of empty bins at the end, what is $\mathbb{E}(E)$?
If E_j is the indicator function for the event that the jth bin is empty, then $E = \sum_{j=1}^{m} E_j$. Are the E_j independent?
Show that

$$\mathbb{P}(|E - \mathbb{E}(E)| \geq \lambda\sqrt{n}) \leq 2e^{-\frac{\lambda^2}{2}},$$

and thus conclude that E is tightly concentrated about it's mean if $m = n$.
(Hint: Let Z_i be the bin into which the ith ball is thrown and consider the martingale sequence $X_i = \mathbb{E}(E \mid Z_1, \ldots, Z_i)$.)

Problem 22. Let $Z_1, \ldots, Z_n$ be a sequence of mutually independent random variables and let $f : \mathcal{Z}^n \to \mathbb{R}$ be a function. Show that the sequence

$$X_i = \mathbb{E}(f(Z_1, \ldots, Z_n) \mid Z_1, \ldots, Z_i)$$

is a martingale.

Problem 23. Suppose we are given n vectors $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^d$ with $||v_i|| \leq 1$ for all i, and we pick a sequence $\epsilon_i \in \{-1, 1\}$ for $1 \leq i \leq n$ independently and uniformly at random. We consider the vector given by

$$v = \sum_{i=1}^{n} \epsilon_i v_i,$$

and let f be the random variable given by $||v||$. Show that changing the value of a single ϵ_i can change the value of f by at most 2.
Show that

$$\mathbb{P}(|f - \mathbb{E}(f)| \geq \lambda\sqrt{n}) \leq 2e^{-\frac{\lambda^2}{2}}.$$

(*Optional* : Can we say anything about $\mathbb{E}(f)$?)

Problem 24. Let Q_n be the n-dimensional hypercube . Given a set $A \subset \{0,1\}^n = V(Q_n)$ and $s \in \mathbb{N}$ let $B(A, s)$ be the set of vertices whose Hamming distance is less than s to A, that is, for every $x \in B(A, s)$ there exists a $y \in A$ such that x and y differ on less than s coordinates.
Let $\epsilon, \lambda > 0$ be such that $e^{-\frac{\lambda^2}{2}} = \epsilon$ and suppose $|A| \geq \epsilon 2^n$. Using Azuma-Hoeffding show that

$$|(B(A, 2\lambda\sqrt{n})| \geq (1 - \epsilon)2^n.$$