Exercise sheet 9

Exercises for the exercise session on 15th January 2025

Problem 29. Given a set $V = \{v_1, v_2, \ldots, v_n\}$ and a collection $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ of subsets of V, $\mathcal{P} \subseteq \mathcal{S}$ is called a *cover* if every element of V is contained in at least one set in $\mathcal{P}$. Suppose each set $S \in \mathcal{S}$ comes with a weight w_S and we wish to find the minimal weight w^* of a cover $\mathcal{P}$, where $w(\mathcal{P}) = \sum_{P \in \mathcal{P}} w_P$.
Describe an algorithm which computes a cover of weight $O(w^* \log n)$ whose expected runtime is polynomial.

Definition. A random variable $X : \Omega \to \mathbb{R}$ on a product space $\Omega = \prod \Omega_i$ is c-*Lipschitz* if changing just one co-ordinate can change the value of X by at most c. Given some function $f : \mathbb{N} \to \mathbb{N}$ we say that X *is f-certifiable* if whenever $X(\omega_1, \omega_2, \ldots, \omega_n) \geq s$ there is a subset $I \subset [n]$ of size $|I| = f(s)$ such that X is greater than s on the entire subspace

$$\{(\omega'_1, \omega'_2, \ldots, \omega'_n) : \omega'_i = \omega_i \text{ for all } i \in I\}.$$

Problem 30. Let X be a c-Lipschitz random variable on a product space which is f-certifiable and let m be the median of X. Then for any $t \geq 0$

$$\mathbb{P}(X \leq m - t) \leq 2e^{-\frac{t^2}{4c^2 f(m)}} \text{ and } \mathbb{P}(X \geq m + t) \leq 2e^{-\frac{t^2}{4c^2 f(m+t)}}.$$

(Hint) : Consider the sets

$$A = \{\omega : X(\omega) \leq m - t\} \text{ and } B = \{\omega : X(\omega) \geq m\}.$$

Problem 31. Let $c, r > 0$. Let X be a non-negative c-Lipschitz random variable on a product space which is f-certifiable with $f(s) = rs$ and let m be the median of X. Then

$$|\mathbb{E}(X) - m| \leq 20c\sqrt{rm} \leq 40c\sqrt{\mathbb{E}(X)}.$$

Problem 32. Let π be a random permutation of $[n]$, let X be the length of the longest increasing subsequence of π and let m be the median of X.

(a) Show that $\frac{1}{3}\sqrt{n} \leq m \leq 3\sqrt{n}$.

(b) Show that we can generate π by choosing an independent sequence $x_1, \ldots, x_n$ uniformly in $[0, 1]$ and letting $\pi(i)$ be the index j of the ith largest x_j, and hence view X as living on a product space.

(c) Show that X is 1-Lipschitz over this product space, and that X is f-certifiable with $f(s) = s$. Deduce that with high probability $X = (1 + o(1))m$.