Differential Operators on Graphs and Waveguides
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Spectral geometry in a rotating frame: properties of the ground state

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joint work with Pavel Exner

We consider the spectral properties of the operator (formally) defined by

$$H_\omega(x_0, y_0) = -\Delta + i\omega\left((x - x_0)\partial_y - (y - y_0)\partial_x\right)$$

on $\Omega \subset \mathbb{R}^2$ subject to the Dirichlet boundary conditions. Here

$$\omega > 0, \quad (x_0, y_0) \in \mathbb{R}^2.$$

Physical motivations

The above operator describes a quantum particle confined to a planar domain Ω rotating around a fixed point with an angular velocity ω. Quantum effects associated with rotation attracted a particular attention in connection with properties of ultracold gases.

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Associated quadratic form

For any $u \in C_0^\infty(\Omega)$ one has

$$(H_\omega(x_0, y_0)u, u)_{L^2(\Omega)} =$$

$$\int_\Omega \left| i\nabla u + \widehat{A}u \right|^2 \mathrm{d}x\,\mathrm{d}y - \frac{\omega^2}{4} \int_\Omega ((x - x_0)^2 + (y - y_0)^2)|u|^2 \,\mathrm{d}x\,\mathrm{d}y,$$

where $\widehat{A} = (-y + y_0, x - x_0)$.

Boundedness of Ω implies that the corresponding operator is bounded from below, hence it allows for Friedrichs extension

$$\widehat{H}_\omega(x_0, y_0) = \left(i\nabla + \frac{\omega}{2}\widehat{A}\right)^2 - \frac{\omega^2}{4}\left((x - x_0)^2 + (y - y_0)^2\right)$$

with the domain $\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$.

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Remark

By simple gauge transformation, namely

$$u(x,y) \mapsto u(x,y)\mathrm{e}^{-i\omega(xy_0-yx_0)/2},$$

the operator $\widehat{H}_\omega(x_0,y_0)$ is unitarily equivalent to

$$\widetilde{H}_\omega(x_0,y_0) = \left(i\nabla + \frac{\omega}{2}A\right)^2 - \frac{\omega^2}{4}\left((x-x_0)^2 + (y-y_0)^2\right)$$

with $A := (-y, x)$.

The spectrum of $\widetilde{H}_\omega(x_0, y_0)$ is purely discrete.

The main object of interest in the talk

Our concern will be **the principal eigenvalue** $\lambda_1^\omega(x_0, y_0)$ of $\widetilde{H}_\omega(x_0, y_0)$.

The first problem concerns

$$(x_0, y_0) \mapsto \lambda_1^\omega(x_0, y_0)$$

for **fixed** Ω and ω, in particular, the existence of its extrema.

Theorem (B.-Exner, 2019)

$\lambda_1^\omega(\cdot,\cdot)$ as a map $\mathbb{R}^2 \to \mathbb{R}$ has no minima. It has a unique maximum.

Remark

$\lambda_1^\omega(x_0, y_0) \to -\infty$ holds as $(x_0, y_0) \to \infty$.
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Sketch of the proof

Let (x_0, y_0) be a possible extrema point.

Step 1

We employ normalized eigenfunctions $u_\omega^{(x_0,y_0)}$ and $v_\omega^{(x_0,y_0)}$ corresponding to $\lambda_1^\omega(x_0, y_0)$ such that

$$u_\omega^{(x_0+t,y_0)} = u_\omega^{(x_0,y_0)} + \mathcal{O}(t),$$
$$v_\omega^{(x_0,y_0+s)} = v_\omega^{(x_0,y_0)} + \mathcal{O}(s),$$

for small values of t and s, where the error term is understood in the L^∞ sense.

The existence of such eigenfunctions is due to

[N. Raymond: Bound States of the Magnetic Schrödinger Operators, EMS, 2017]

If the eigenvalue $\lambda_1^\omega(x_0, y_0)$ is simple then $u_\omega^{(x_0,y_0)} = v_\omega^{(x_0,y_0)}$. In fact we shall see that this not true in general.

Step 2

The key point – to prove the following implication:

$$(x_0, y_0) \text{ is an extremum point}$$

$$\Downarrow$$

$$\overbrace{\int_\Omega (x - x_0)\,|u_\omega^{(x_0,y_0)}|^2\,\mathrm{d}x\,\mathrm{d}y = 0 \quad \text{and} \quad \int_\Omega (y - y_0)\,|v_\omega^{(x_0,y_0)}|^2\,\mathrm{d}x\,\mathrm{d}y = 0}$$

Idea of its proof: assume that this in not true, for example, one has

$$\int_\Omega (x - x_0)\,|u_\omega^{(x_0,y_0)}|^2\,\mathrm{d}x\,\mathrm{d}y > 0.$$

Using min-max principle, one can show that the above inequality implies for any $h < 0$ small enough $\lambda_1^\omega(x_0 + h, y_0) < \lambda_1^\omega(x_0, y_0)$.

Also, using min-max principle, one can deduce for small $t > 0$ $\lambda_1^\omega(x_0, y_0) < \lambda_1^\omega(x_0 + t, y_0)$.

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Step 3

Using (cf. Step 2)

$$\int_\Omega (x - x_0)\,|u_\omega^{(x_0,y_0)}|^2\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \int_\Omega (y - y_0)\,|v_\omega^{(x_0,y_0)}|^2\,\mathrm{d}x\,\mathrm{d}y = 0$$

and min-max principle one can prove that for all nonzero and sufficiently small h

$$\lambda_1^\omega(x_0 + h, y_0) < \lambda_1^\omega(x_0, y_0).$$

Thus (x_0, y_0) is a point of maximum.

Theorem (B.-Exner, 2019)

Let Ω be convex, then

$$(x_0, y_0) \mapsto \lambda_1^\omega(x_0, y_0)$$

reaches its maximum at a point belonging to Ω.

If ω is **small** then the position of the maximum can be described more precisely.

Definition

Given a region $\Sigma \subset \mathbb{R}^2$ and a line P, we denote by Σ^P the mirror image of Σ with respect to P.

Theorem (B.-Exner, 2019)

Let Ω be convex set and P be a line which divides Ω into two parts, Ω_1 and Ω_2, in such a way that $\Omega_1^P \subset \Omega_2$. Then for small enough values of ω the point at which $\lambda_1^\omega(x_0, y_0)$ attains its maximum does not belong to Ω_1.

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Sketch of the proof

Recall: $H_\omega(x_0, y_0) = -\Delta_D^\Omega + i\omega((x - x_0)\partial_y - (y - y_0)\partial_x)$.

Without loss of generality we may suppose that P is parallel to the Y axis. Let $(x_0, y_0) \in \Omega_1$ and assume that P passes through it.

Consider first the case $\omega = 0$. Let u_D be the ground state eigenfunction of the Dirichlet Laplacian,

$$-\Delta_\Omega^D u_D = \lambda_1^D u_D.$$

In view of standard perturbation theory for all sufficiently small ω the ground state eigenvalue $\lambda_1^\omega(x_0, y_0)$ is simple and the corresponding eigenfunction satisfies

$$u^\omega(x_0, y_0)(x, y) = u_D(x, y) + \mathcal{O}(\omega).$$

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Key lemma

$$\int_{\Omega}(x-x_0)(u_D(x,y))^2\,\mathrm{d}x\,\mathrm{d}y > 0.$$

Proof: later.

Using the above lemma we conclude (since $\|u^{\omega}(x_0,y_0)-u_D\|_{L^\infty} \ll 1$)

$$\int_{\Omega}(x-x_0)(u^{\omega}(x_0,y_0)(x,y))^2\,\mathrm{d}x\,\mathrm{d}y > 0.$$

But this contradicts to the neccesary condition for the point (x_0,y_0) to be a point of maximum.

Recall

The necessary condition for the maximum is $\int_{\Omega}(x-x_0)(u^{\omega}(x_0,y_0)(x,y))^2\,\mathrm{d}x\,\mathrm{d}y = 0$.

Thus (x_0,y_0) is not a point of maximum. It remains to prove the above lemma.

Proof of Lemma

$$v(x,y) := u_D(x,y) - u_D(x^P,y) \quad \text{on} \quad \Omega_1,$$

where (x^P, y) is the mirror image of (x, y) with respect to P.

Positivity of u_D implies

$$v|_{\partial\Omega_1} \leq 0.$$

- $v|_{\partial\Omega_1} \leq 0$
- $-\Delta v = \lambda_1^D v \quad \text{on} \quad \Omega_1$
- the maximum principle for the second order elliptic partial differential equations

$$\Downarrow$$

$$v < 0 \quad \text{on} \quad \Omega_1$$

$$\Updownarrow$$

$$u_D(x, y) \leq u_D(x^P, y), \quad (x, y) \in \Omega_1 .$$

$$\Downarrow_{\Omega_1^P \subset \Omega_2}$$

$$\int_\Omega (x - x_0)(u_D(x, y))^2 \, \mathrm{d}x \, \mathrm{d}y > 0$$

Our next topic is to compare the ground state eigenvalue of $H_\omega(x_0, y_0)$ with **different values** of ω.

Theorem (B.-Exner, 2019)

$$\lambda_1^\omega(x_0, y_0) \leq \lambda_1^D(\Omega),$$

where $\lambda_1^D(\Omega)$ is the ground state eigenvalue of the Dirichlet Laplacian $-\Delta_D^\Omega$ on Ω.

Moreover, the inequality is sharp for $\omega > 0$ provided the region Ω does not have full rotational symmetry (disk or a circular annulus with (x_0, y_0) being its center).

Let us now look more closely at the situation when the system has a rotational symmetry.

Hereafter in this subsection Ω is a **disk** of radius R rotating around its center which we identify with the point $(0,0)$.

In this case the spectrum is

$$\lambda_{m,k}(R,\omega) = \frac{j_{m,k}^2}{R^2} - m\omega, \quad m \in \mathbb{Z},\ k \in \mathbb{N},$$

where $j_{m,k}$ is the kth positive zero of Bessel function of the first kind J_m.

$$\lambda_1^\omega(x_0,y_0) = \inf_{m,k}\lambda_{m,k} = \inf_{m\geq 0}\lambda_{m,1}$$

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Lemma

For each $m \in \mathbb{N}$ there is positive $\omega_0 > 0$ such that

$$\lambda_{m,1} \geq \frac{j_{0,1}^2}{R^2}, \quad \omega \leq \omega_0.$$

Remark

$$\frac{j_{0,1}^2}{R^2} = \lambda_1^D(\Omega).$$

Theorem (B.-Exner, 2019)

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Corollary (Theorem + Lemma + Remark)

$$\lambda_1^\omega(x_0, y_0) = \lambda_1^D(\Omega), \quad \omega \in (0, \omega_0].$$

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In the last part of the talk we demonstrate an estimate in which the ground state eigenvalue is compared to that of a disk of the same area.

For this purpose we add the index specifying the region writing $\widetilde{H}_{\omega,\Omega}(x_0, y_0)$ and $\lambda^{\omega}_{1,\Omega}(x_0, y_0)$.

We restrict our attention to convex regions with a fixed $(x_0, y_0) \in \Omega$ which we can write as

$$\Omega = \{(x_0 + r\cos\varphi, y_0 + r\sin\varphi) : \varphi \in [0, 2\pi), \ r \in [0, R(\varphi))\}$$

for a suitable 2π-periodic function R.

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Theorem (B.-Exner, 2019)

Suppose that $\pi R_0^2 = |\Omega|$ and denote by B the disk of radius R_0 and center in the origin, then

$$\lambda_{1,\Omega}^{\omega}(x_0, y_0) \leq \lambda_{1,B}^{\omega}(0,0)$$

$$+ \left(\int_0^{2\pi} \left(\frac{R'(\varphi)}{R(\varphi)} \right)^2 \mathrm{d}\varphi \right) \sup_{0 \leq m \leq \frac{R_0^2\omega + \sqrt{R_0^4\omega^2 + 4j_{0,1}^2}}{2}} \frac{j_{m,1}^2 - m^2}{2\pi R_0^2}.$$

For large values of ω the right-hand -side behaves as

$$\lambda_{1,B}^{\omega}(0,0) + \mathcal{O}\left(R_0^{2/3}\omega^{4/3}\right) \to -\infty \quad \textit{as} \quad \omega \to \infty$$

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Thank you for your attention