Periodic quantum graphs in magnetic fields

Differential Operators on Graphs and Waveguides – Graz 2019

Simon Becker

Cambridge

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Brüning–Geyler–Pankrashkin '07

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Theorem (Helffer-Sjöstrand+ B.-Han-Jitomirskaya)

The spectrum of $Q^h_{\square}$ *or* $Q^h_{\hexagon}$ *for* $\frac{h}{2\pi} \in \mathbb{Q}$ *is band spectrum and a.c..* *If* $\frac{h}{2\pi} \in \mathbb{R}\backslash\mathbb{Q}$ *the spectrum is a Cantor set (closed, nowhere dense, no isolated points) of Lebesgue measure zero and s.c..*

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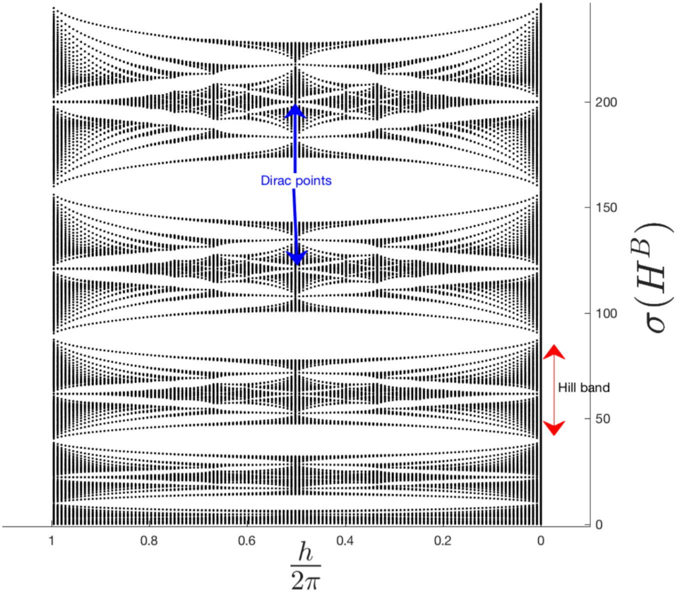

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The determinant of the symbol $Q_{\hexagon}$ is given by $-|1+e^{ix}+e^{i\xi}|^2/9$, and it vanishes at

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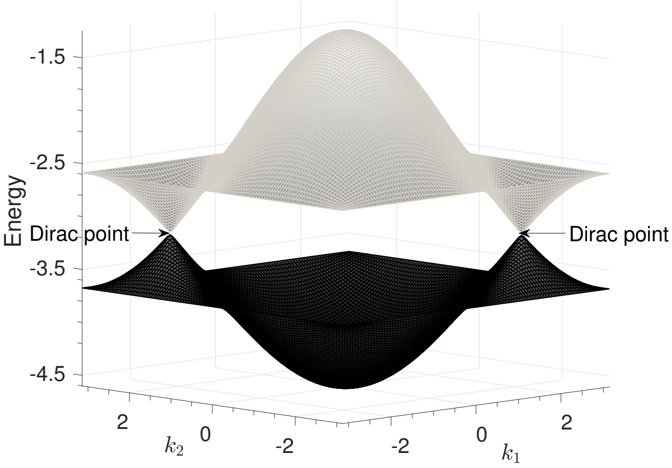
and find that

$$\begin{aligned} 1+e^{ix}+e^{i\xi} &= c(\eta \mp iy) + \mathcal{O}(y^2+\eta^2), \\ 1+e^{-ix}+e^{-i\xi} &= c(\eta \pm iy) + \mathcal{O}(y^2+\eta^2), \end{aligned}$$

where $c = 3^{\frac{1}{4}}2^{-\frac{1}{2}}$ by choosing $a = \pm 2^{-\frac{3}{4}}3^{-\frac{1}{4}}$ and $b = \pm 2^{-\frac{1}{4}}3^{\frac{1}{4}}$.

This is a plot of the first two bands of the spectrum of $H^{B=0}$ on the hexagonal lattice (cf. Kuchment-Post):



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$$\int f(E)d\rho(E) = \frac{h}{\pi\,|b_1 \wedge b_2|} \sum_{n\in\mathbb{Z}} f(E_n(h)) + \mathcal{O}_{\|f\|_{C^\alpha}}(h^\infty)$$

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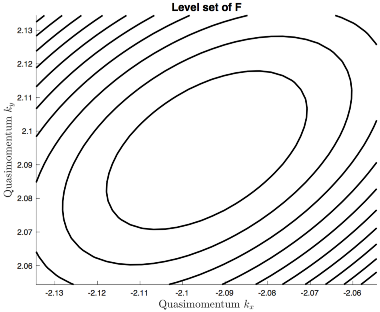

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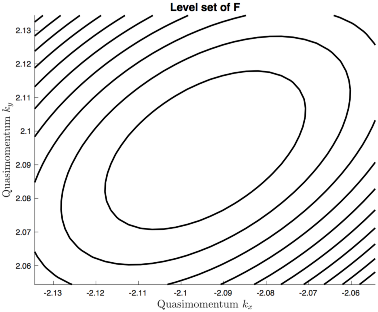

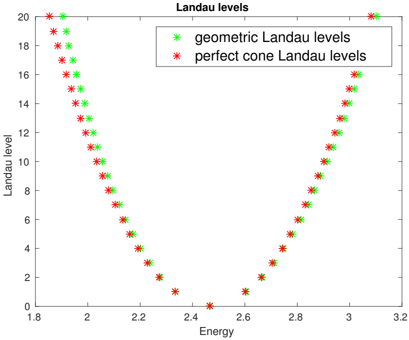

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For operators $A \in \mathcal{L}(\ell^2(\mathbb{Z}^2, \mathbb{C}^n))$ given by

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Quantum Hall effect

For projections P, Q such that $P - Q$ is compact we define

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- *Avron, Seiler, Simon*: $(U_a\psi)(x) := e^{-i\theta_a(x)}\psi(x)$ with $\theta_a(x) := \arg(x - a) \in (-\pi, \pi]$.

$$\sigma = \frac{1}{2\pi} \operatorname{ind}(P, U_a P U_a^*) = \frac{1}{2\pi} \operatorname{tr}(P - U_a P U_a^*)^3.$$

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- Use results on spectral theory to conclude the existence of large spectral gaps between Landau levels.
- Use results from non-commutative geometry:

On $\ell^2(\mathbb{Z}^2)$ we define the rotation algebra $\mathcal{A}_\hbar$ as the operator norm closure of

$$A_\hbar := \left\{ T \in \mathcal{L}(\ell^2(\mathbb{Z}^2;\mathbb{C}^n)); \exists n \in \mathbb{N},\ c_\gamma \in \mathbb{C} : T = \sum_{|\gamma|\le n} c_\gamma \tau_\gamma^h \right\}$$

where $\tau_\delta^h(f)(\gamma) := e^{-i\frac{h}{2}\sigma_{\text{symp}}(\gamma,\delta)} f(\gamma-\delta)$.

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where $\tau_\delta^h(f)(\gamma) := e^{-i\frac{h}{2}\sigma_{\text{symp}}(\gamma,\delta)} f(\gamma-\delta)$. From results by Voiculescu-Pimser and Rieffel it follows that for any projection $P \in \mathcal{A}_\hbar$

$$\widehat{\operatorname{tr}}(P) = \gamma_1 \widehat{\operatorname{tr}}(\mathrm{id}) + \gamma_2 \widehat{\operatorname{tr}}(\mathrm{P}_R) = \gamma_1 + \gamma_2 \frac{h}{2\pi} \in \left(\mathbb{Z} + \frac{h}{2\pi} \bmod 1\ \mathbb{Z}\right) \cap [0,1]$$

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Combining this with the semiclassical analysis shows that

$$\sigma(H_{\square}^B) = \tfrac{n}{2\pi},\, n \ge 1$$
$$\sigma(H_{\hexagon}^B) = \begin{cases} \frac{2n+1}{2\pi} & n \ge 0 \\ \frac{2n-1}{2\pi} & n < 0. \end{cases}$$

The proof of delocalization is then based on the following ideas (cf. Germinet, Klein, Schenker):

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- Show that the Hall conductivity is constant in regions of strong dynamical localization.
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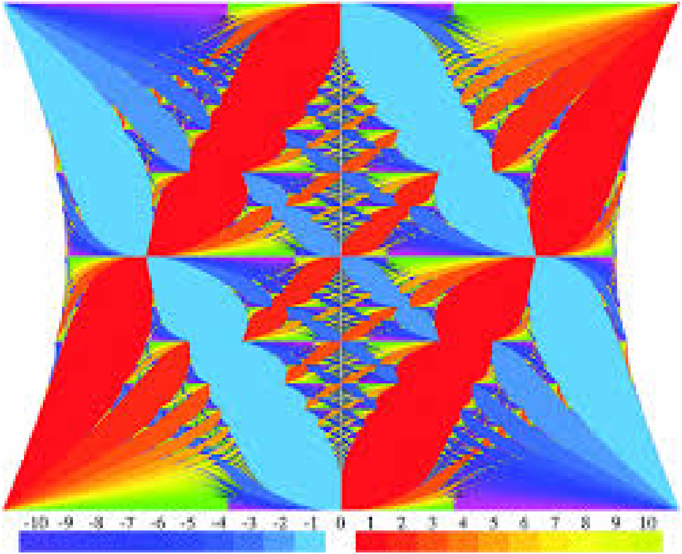
We conclude:

Theorem

Between each of the Landau levels of the random Schrödinger operator $H^B_{\lambda,\omega}$ with B and λ sufficiently small there exists a mobility edge, i.e. an energy at which delocalization occurs.


-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10

Dynamical delocalization is characterized in terms of

$$M^h_{\lambda,\omega}(p,\zeta,t) = \left\| \langle x \rangle^{p/2} e^{-itH^h_{\lambda,\omega}} \zeta(H^h_{\lambda,\omega}) \delta_0 \right\|^2_{\mathrm{HS}}$$

where $\zeta \in C^\infty_{c,+}(\mathbb{R})$. We also consider the averaged expression

$$M^h_\lambda(p,\zeta,T) = \frac{1}{T} \int_0^\infty \mathbb{E}\left(M^h_{\lambda,\omega}(p,\zeta,t) \right) e^{-t/T} \, dt.$$

The *(lower) transport exponent* is defined by

$$\beta^h_\lambda(p,\zeta) = \liminf_{T\to\infty} \frac{\log_+ M^h_\lambda(p,\zeta,T)}{p \log(T)}, \text{ for } p > 0, \zeta \in C^\infty_{c,+}(\mathbb{R})$$

and from this one defines the *p-th local transport exponent*

$$\beta^h_\lambda(p,E) = \inf_{I \ni E} \sup_{\zeta \in C^\infty_{c,+}(I)} \beta^h_\lambda(p,\zeta) \in [0,1].$$

The local lower transport exponent is then defined as

$$\beta^h_\lambda(E) := \sup_{p>0} \beta^h_\lambda(p,E).$$

The exponent $\beta^h_\lambda(E)$ is a measure of transport associated with the energy E.

One then defines two complementary regions, the (relatively open) *region of dynamical localization or insulator region*

$$\Sigma_{\lambda}^{h,\mathrm{DL}} = \left\{ E \in \mathbb{R}; \beta_{\lambda}^{h}(E) = 0 \right\} \tag{1}$$

and the (relatively closed) *region of dynamical delocalization or metallic transport*

$$\Sigma_{\lambda}^{h,\mathrm{DD}} = \left\{ E \in \mathbb{R}; \beta_{\lambda}^{h}(E) > 0 \right\}. \tag{2}$$