A new complex frequency spectrum for the analysis of transmission efficiency in waveguide-like geometries

Anne-Sophie Bonnet-Ben Dhia[1]
Lucas Chesnel[2] Vincent Pagneux[3]

[1] *POEMS (CNRS-ENSTA-INRIA), Palaiseau, France*
[2] *Equipe DEFI (INRIA, CMAP-X), Palaiseau, France.*
[3] *LAUM (CNRS, Université du Maine), Le Mans, France*

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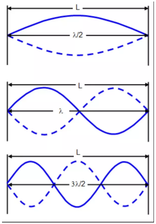
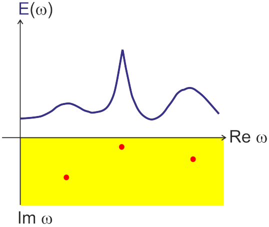
Spectral theory and wave phenomena

The spectral theory is classically used to study resonance phenomena:

- eigenfrequencies of a string, a closed acoustic cavity, etc...
- complex resonances of "open" cavities (with leakage)

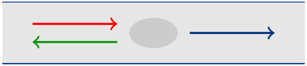




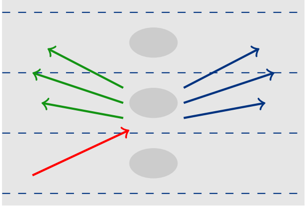
A new point of view: find similar spectral approaches to quantify the efficiency of the transmission phenomena.

This notion of transmission appears naturally in devices involving waveguides or gratings (intensively used in optics and acoustics).

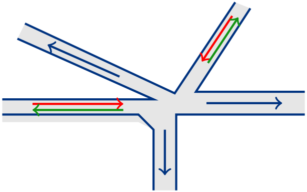
Some typical devices



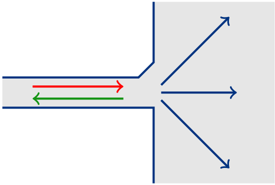
Perturbed waveguide



Grating



Junction of waveguides

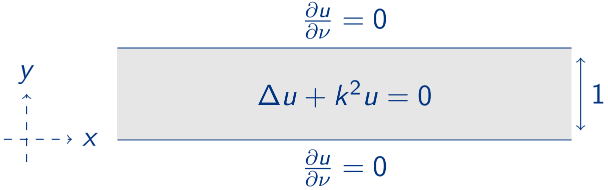


Baffled radiating waveguide

A usual objective is to get a perfect transmission without any reflection.

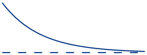
Time-harmonic scattering in waveguide

The acoustic waveguide: $\Omega = \mathbb{R} \times (0,1)$, $k = \omega/c$, $e^{-i\omega t}$

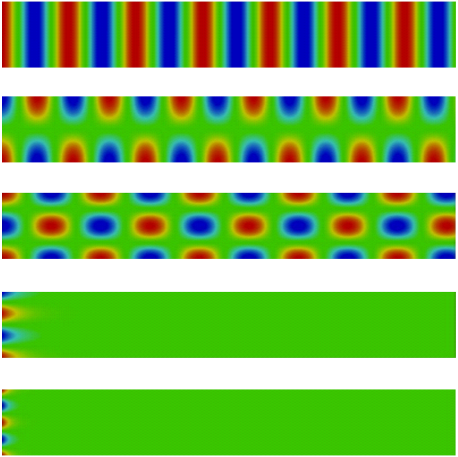


- A finite number of propagating modes for $k > n\pi$:
  $u_n^{\pm}(x,y) = \cos(n\pi y) e^{\pm i\beta_n x} \quad \beta_n = \sqrt{k^2 - n^2\pi^2}$
  ($+/-$ correspond to right/left going modes)

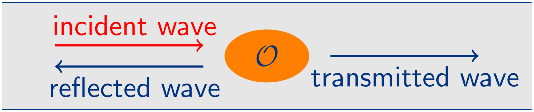
- An infinity of evanescent modes for $k < n\pi$:
  $u_n^{\pm}(x,y) = \cos(n\pi y) e^{\mp \gamma_n x} \quad \gamma_n = \sqrt{n^2\pi^2 - k^2}$



An example with 3 propagating modes:



$\mathcal{O} \subset \Omega$
$\inf(1+\rho) > 0$
$\mathrm{supp}(\rho) \subset \mathcal{O}$



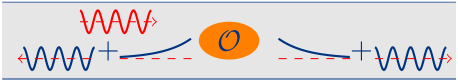
- The total field $u = u_{inc} + u_{sca}$ satisfies the equations

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega)$$

- The incident wave is a superposition of propagating modes:

$$u_{inc} = \sum_{n=0}^{N_P} a_n u_n^+$$

- The scattered field u_{sca} is outgoing:

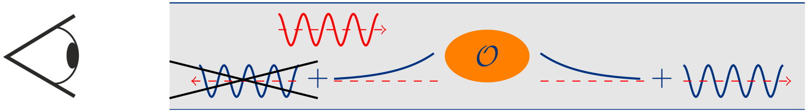


No-reflection

At particular frequencies k, it occurs that, for some u_{inc},

$$x \to -\infty \quad u_{sca} \to 0$$

We say that the obstacle $\mathcal{O}$ produces no reflection. The wave is totally transmitted. And the obstacle is invisible for an observer located far at the left-hand side.



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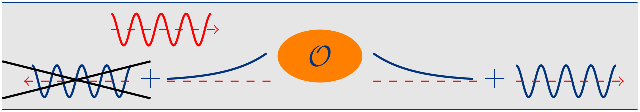
$k \in \mathscr{K}$



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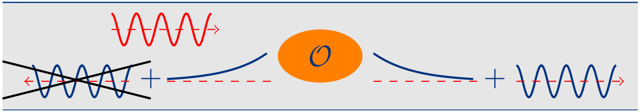
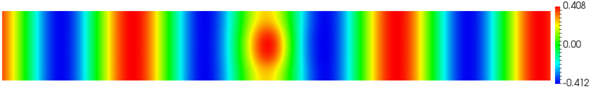


OBJECTIVE

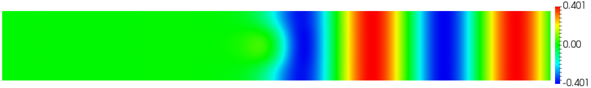
Find a way to compute directly the set $\mathscr{K}$ of no-reflection frequencies by solving an eigenvalue problem.

An illustration of no-reflection phenomenon

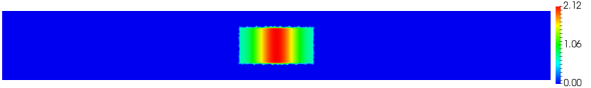
Incident field $u_{inc} = e^{ikx}$



Total field u



Scattered field u_{sca}



Perturbation ρ

The main idea

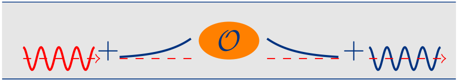
The total field u always satisfies the homogeneous equations:

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega)$$

where k^2 plays the role of an eigenvalue.

No-reflection modes ($k \in \mathscr{K}$)

The total field of the scattering problem u is ingoing at the left-hand side of $\mathcal{O}$ and outgoing at the right-hand side of $\mathcal{O}$.



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No-reflection modes ($k \in \mathscr{K}$) New!

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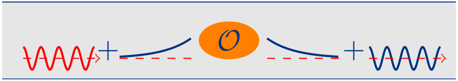
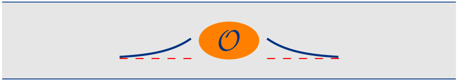


Trapped modes ($k \in \mathscr{T}$) Classical!

The total field $u \in L^2(\Omega)$.



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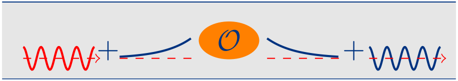
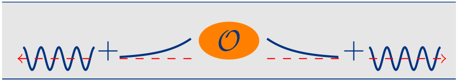


Trapped modes ($k \in \mathscr{T}$) Classical!

The total field u is outgoing on both sides of the obstacle $\mathcal{O}$.

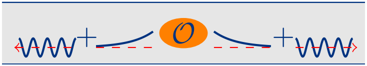


The main idea

For both problems, the idea is to use a complex scaling at both sides of the obstacle, so that propagating waves become evanescent.

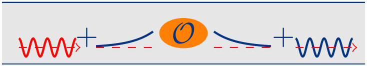
Trapped modes

$k \in \mathscr{T}$: u is outgoing on both sides of $\mathcal{O}$.



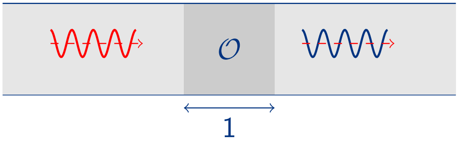
No-reflection modes

$k \in \mathscr{K}$: u is ingoing (resp. outgoing) at the left (resp. right) of $\mathcal{O}$.



The novelty

To compute the no-reflection frequencies, use a complex scaling with complex conjugate parameters at both sides of the obstacle



The 1D case has been studied with a spectral point of view in:

H. Hernandez-Coronado, D. Krejcirik and P. Siegl,
Perfect transmission scattering as a $\mathcal{PT}$-symmetric spectral problem,
Physics Letters A (2011).

Our approach allows us to extend some of their results to higher dimensions.

An additional complexity comes from the presence of evanescent modes.

Outline

1. Spectrum of trapped modes frequencies
2. Spectrum of no-reflection frequencies
3. Extensions to other configurations

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2. Spectrum of no-reflection frequencies
3. Extensions to other configurations

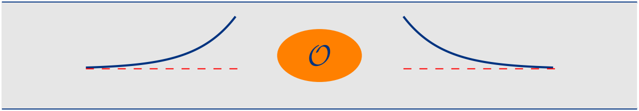
The spectral problem for trapped modes

Definition

A trapped mode of the perturbed waveguide is a solution $u \neq 0$ of

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega)$$

such that $u \in L^2(\Omega)$.



- There is a huge literature on trapped modes: Davies, Evans, Exner, Levitin, McIver, Nazarov, Vassiliev, ...
- Existence of trapped modes is proved in specific configurations (for instance symmetric with respect to the horizontal mid-axis) (Evans, Levitin and Vassiliev)

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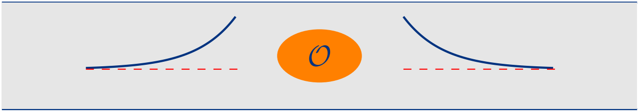

Let us consider the following unbounded operator of $L^2(\Omega)$:

$$D(A) = \{u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\} \qquad Au = -\frac{1}{1+\rho}\Delta u$$

$$\Delta u + k^2(1+\rho)u = 0 \iff Au = k^2 u$$

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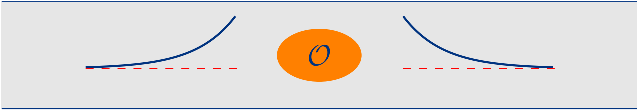

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The trapped modes ($k \in \mathscr{T}$) correspond to real eigenvalues k^2 of A.

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For the scalar product of $L^2(\Omega)$ with weight $1+\rho$:

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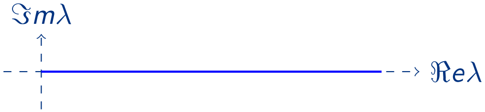
SPECTRAL FEATURES OF A

- A is a positive self-adjoint operator of $L^2(\Omega)$.
- $\sigma(A) = \sigma_{ess}(A) = \mathbb{R}^+$ and $\sigma_{disc}(A) = \emptyset$



THE SPECTRAL PROBLEM FOR TRAPPED MODES

Trapped modes ($k \in \mathscr{T}$) correspond to real eigenvalues k^2 of

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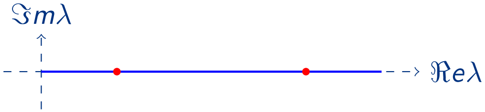
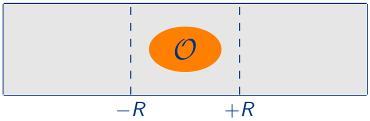
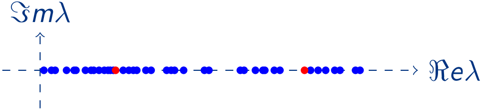
SPECTRAL FEATURES OF A

- A is a positive self-adjoint operator of $L^2(\Omega)$.
- $\sigma(A) = \sigma_{ess}(A) = \mathbb{R}^+$ and $\sigma_{disc}(A) = \emptyset$
- Trapped modes are embedded eigenvalues of A !



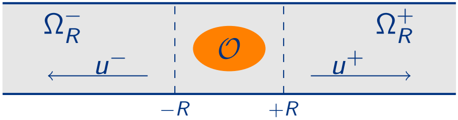
Problem: a direct Finite Element computation in a large bounded domain produces spurious eigenvalues!





Solution: the complex scaling (Aguilar, Balslev, Combes, Simon 70)

A main tool: the complex scaling

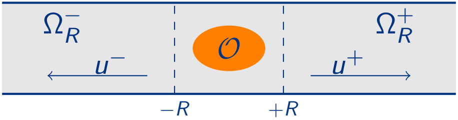


The magic idea:

1. consider the second caracterization of trapped modes: $u^\pm$ outgoing,
2. apply a complex scaling to $u^\pm$ in the x direction:

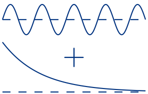
$$u_\alpha^\pm(x,y) = u^\pm\left(\pm R + \frac{x \mp R}{\alpha}, y\right) \text{ for } (x,y) \in \Omega_R^\pm$$

One can chose $\alpha \in \mathbb{C}$ such that $u_\alpha^\pm \in L^2(\Omega_R^\pm)$!

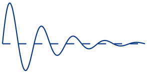


If $\alpha = e^{-i\theta}$ with $0 < \theta < \pi/2$, propagating modes become evanescent:

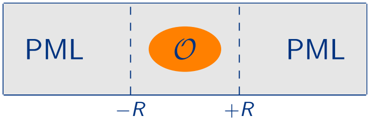
$$u^+(x,y) = \sum_{n \le N_P} a_n \cos(n\pi y) e^{i\sqrt{k^2-n^2\pi^2}(x-R)} + \sum_{n > N_P} a_n \cos(n\pi y) e^{-\sqrt{n^2\pi^2-k^2}(x-R)}$$



$$u_\alpha^+(x,y) = \sum_{n \le N_P} a_n \cos(n\pi y) e^{\frac{i\sqrt{k^2-n^2\pi^2}}{\alpha}(x-R)} + \sum_{n > N_P} a_n \cos(n\pi y) e^{-\frac{\sqrt{n^2\pi^2-k^2}}{\alpha}(x-R)}$$



and the same for u_α^- with the same α.

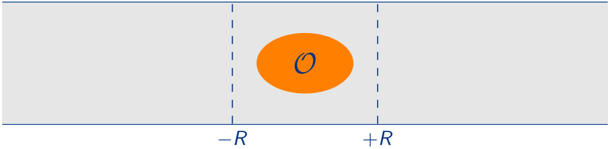


Since $u_\alpha^\pm$ are exponentially decaying at infinity, one can truncate the waveguide for numerical purposes !

This is the celebrated method of Perfectly Matched Layers (see Bécache et al., Kalvin, Lu et al., etc... for scattering in waveguides).

Let us consider now the following unbounded operator:

$$
\begin{aligned}
D(A_\alpha) &= \{u \in L^2(\Omega); A_\alpha u \in L^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\} \\
A_\alpha u &= -\frac{1}{1+\rho(x,y)}\left(\alpha(x)\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right) + \frac{\partial^2 u}{\partial y^2}\right)
\end{aligned}
$$



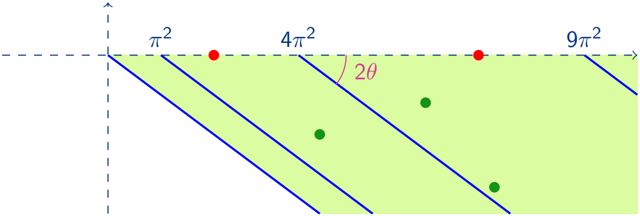
where $\alpha(x) = e^{-i\theta}$ $\alpha(x) = 1$ $\alpha(x) = e^{-i\theta}$

Complex scaling for trapped modes

Spectral features of A_α

- A_α is a non self-adjoint operator.
- $\sigma_{ess}(A_\alpha) = \cup_{n\geq 0}\{n^2\pi^2 + e^{-2i\theta}t^2; t \in \mathbb{R}\}$ *(Weyl sequences)*
- $\sigma(A_\alpha) = \sigma_{ess}(A_\alpha) \dot\cup \sigma_{disc}(A_\alpha)$
- $\sigma(A_\alpha) \subset \{z \in \mathbb{C}; -2\theta < \arg(z) \leq 0\}$

(see Kalvin, Kim and Pasciak)

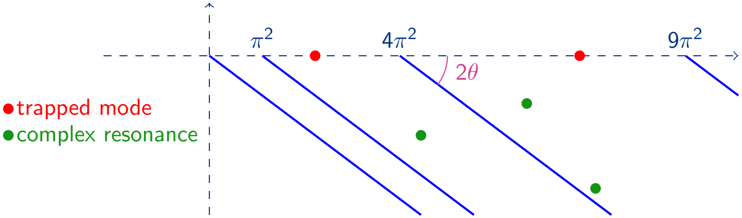


Trapped modes and complex resonances

Discrete spectrum of A_α

- Trapped modes correspond to discrete real eigenvalues of A_α !
- Other eigenvalues correspond to complex resonances, with a field u exponentially growing at infinity.

Spectrum of A_α:



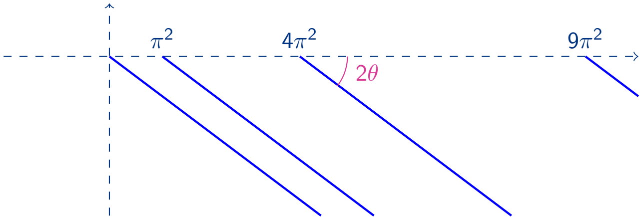
Proof of the second item:

$$\begin{aligned}\sigma_{ess}(A_\alpha) &= \sigma_{ess}(-\Delta_\theta) \\ &= \bigcup_{n\geq 0} \sigma_{ess}(-\Delta_\theta^{(n)}) \\ &= \bigcup_{n\geq 0} \{n^2\pi^2 + e^{-2i\theta}t^2; t \in \mathbb{R}\}\end{aligned}$$

$$\Delta_\theta = e^{-2i\theta}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

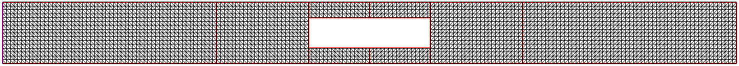
$$\Delta_\theta^{(n)} = e^{-2i\theta}\frac{\partial^2}{\partial x^2} + n^2\pi^2$$

Essential spectrum of A_α:



The numerical results have been obtained by a finite element discretization with FreeFem++.

Here the scatterer is a non-penetrable rectangular obstacle in the middle of the waveguide:



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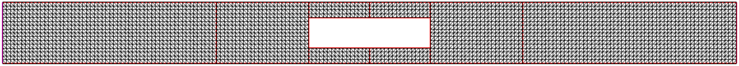
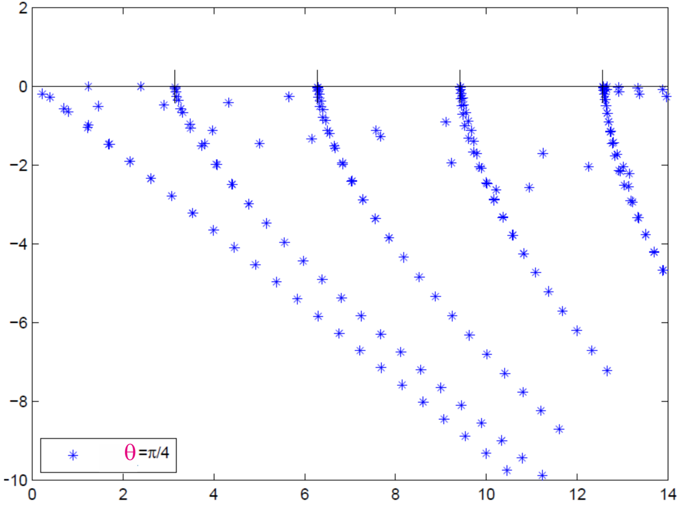


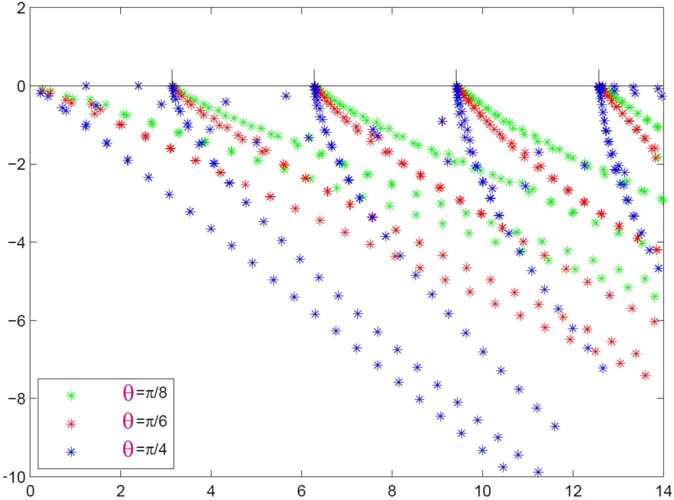
We use a complex scaling in the magenta parts:

In the next slides, we represent the square-root of the spectrum, which corresponds to k values.

Numerical illustration

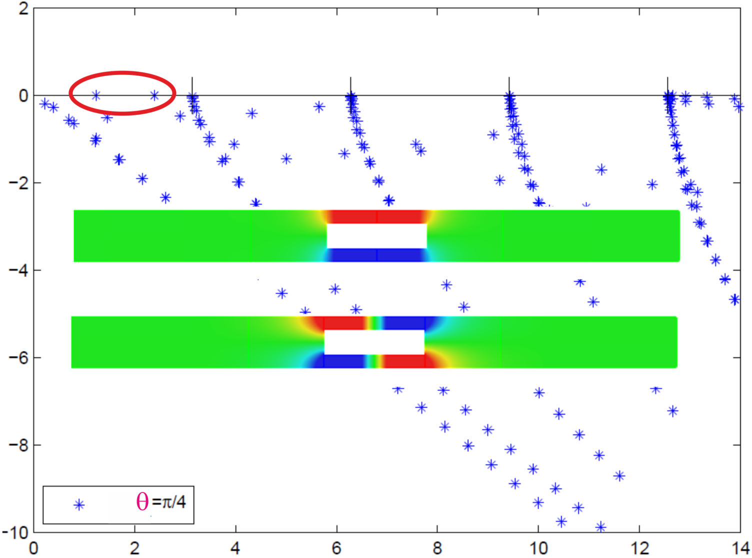


Numerical illustration



Numerical illustration

There are two trapped modes:



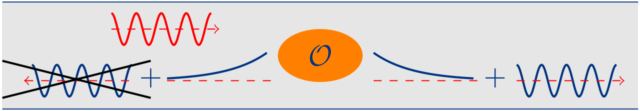
Outline

A new complex spectrum linked to $\mathscr{K}$

with "conjugate" PMLs

A simple and important remark

For $k \in \mathscr{K}$, the total field is ingoing at the left-hand side of $\mathcal{O}$ and outgoing at the right-hand side of $\mathcal{O}$.



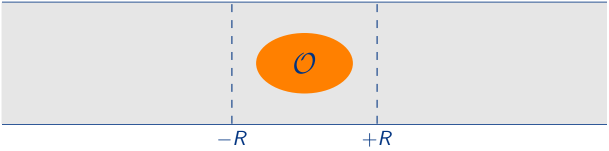
The idea is to use a complex scaling (and numerically PMLs), with complex conjugate parameters at both sides of the obstacle, so that the transformed total field u will belong to $L^2(\Omega)$.

A new complex spectrum linked to $\mathscr{K}$

with "conjugate" PMLs

Let us consider now the following unbounded operator:

$$
\begin{aligned}
D(A_{\tilde{\alpha}}) &= \{u \in L^2(\Omega); A_{\tilde{\alpha}} u \in L^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\} \\
A_{\tilde{\alpha}} u &= -\frac{1}{1+\rho(x,y)} \left(\tilde{\alpha}(x) \frac{\partial}{\partial x} \left(\tilde{\alpha}(x) \frac{\partial u}{\partial x} \right) + \frac{\partial^2 u}{\partial y^2} \right)
\end{aligned}
$$



$\tilde{\alpha}(x) = e^{i\theta}$ $\quad$ $\tilde{\alpha}(x) = 1$ $\quad$ $\tilde{\alpha}(x) = e^{-i\theta}$

A new complex spectrum linked to $\mathscr{K}$

with "conjugate" PMLs

Let us consider now the following unbounded operator:

$$\begin{aligned} D(A_{\tilde{\alpha}}) &= \{u \in L^2(\Omega); A_{\tilde{\alpha}} u \in L^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\} \\ A_{\tilde{\alpha}} u &= -\frac{1}{1+\rho(x,y)} \left(\tilde{\alpha}(x) \frac{\partial}{\partial x} \left(\tilde{\alpha}(x) \frac{\partial u}{\partial x} \right) + \frac{\partial^2 u}{\partial y^2} \right) \end{aligned}$$

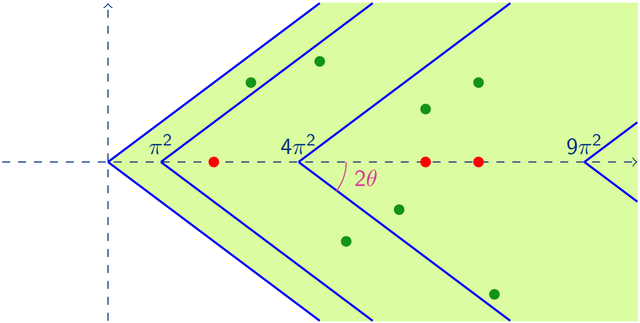
Spectral features of $A_{\tilde{\alpha}}$

- $A_{\tilde{\alpha}}$ is a non self-adjoint operator.
- $\sigma_{ess}(A_{\tilde{\alpha}}) = \bigcup_{n \geq 0} \{n^2\pi^2 + e^{2i\theta} t^2; t \in \mathbb{R}\} \cup \{n^2\pi^2 + e^{-2i\theta} t^2; t \in \mathbb{R}\}$
- $\sigma_{disc}(A_{\tilde{\alpha}}) \subset \{z \in \mathbb{C}; -2\theta < \arg(z) < 2\theta\}$

A NEW COMPLEX SPECTRUM LINKED TO $\mathscr{K}$

WITH "CONJUGATE" PMLS

Typical expected spectrum of $A_{\tilde{\alpha}}$:



SPECTRAL FEATURES OF $A_{\tilde{\alpha}}$

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A new complex spectrum linked to $\mathscr{K}$

with "conjugate" PMLs

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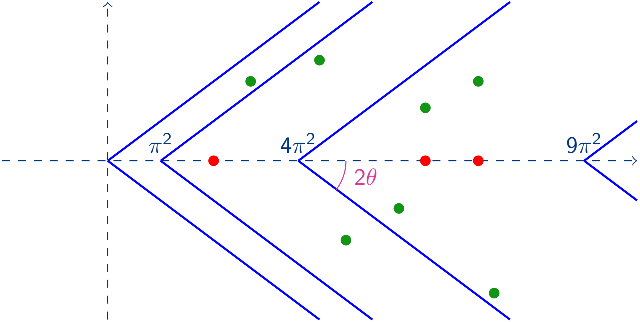
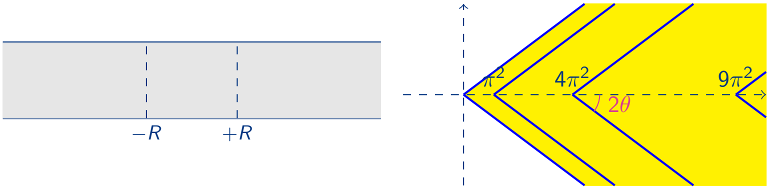


Difficulty: $\mathbb{C} \backslash \sigma_{ess}(A_{\tilde{\alpha}})$ is not a connected set.

Conjecture

$$\sigma(A_{\tilde{\alpha}}) = \sigma_{ess}(A_{\tilde{\alpha}}) \cup \sigma_{disc}(A_{\tilde{\alpha}}) \text{ if } \rho \neq 0$$

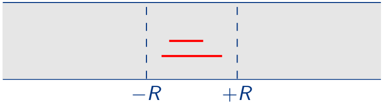
In the unperturbed case ($\rho = 0$):

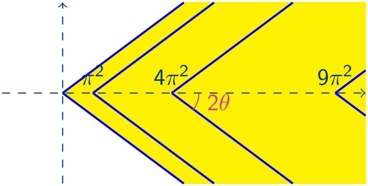


All k^2 in the yellow zone are eigenvalues of $A_{\tilde{\alpha}}$!

Pathological cases

And the same result holds with horizontal cracks !





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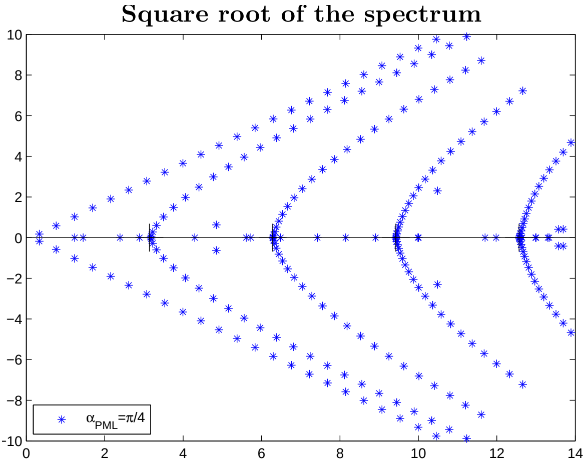
Numerical illustration

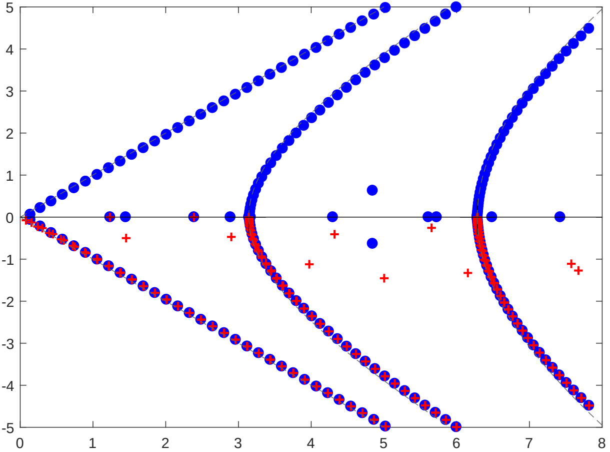
for a rectangular symmetric cavity



- The spectrum is symmetric w.r.t. the real axis ($\mathcal{PT}$-symmetry) .
- There are much more real eigenvalues than for trapped modes.

Numerical illustration

for a rectangular symmetric cavity



In red: classical complex scaling
In blue: conjugate complex scaling

Numerical illustration

for a rectangular symmetric cavity

For $k^2 \in \sigma_{disc}(A_{\tilde{\alpha}}) \cap \mathbb{R}$, the eigenmode is such that:

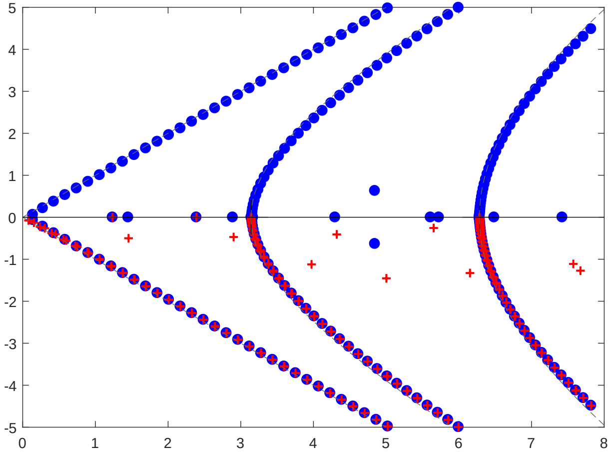
There are two cases:

- Either u contains propagating parts and it is a no-reflection mode: $k \in \mathscr{K}$.
- Either u is evanescent on both sides and it is a trapped mode: $k \in \mathscr{T}$.

Theorem

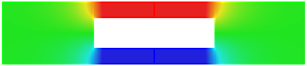
$$\sigma_{disc}(A_{\tilde{\alpha}}) \cap \mathbb{R} = \{k^2 \in \mathbb{R}; k \in \mathscr{K} \cup \mathscr{T}\}$$

Validation



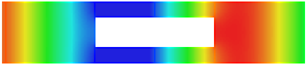
Red: classical PMLs
Blue: conjugate PMLs

Let us focus on the eigenmodes such that $0 < k < \pi$:



First trapped mode:
$k = 1.2355\cdots$

Second trapped mode:
$k = 2.3897\cdots$

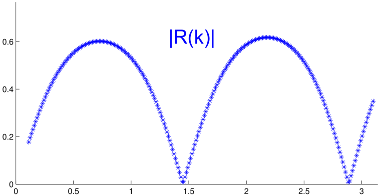


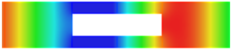
First no-reflection mode:
$k = 1.4513\cdots$

Second no-reflection mode:
$k = 2.8896\cdots$

Validation

To validate this result, we compute the amplitude of the reflected plane wave for $0 < k < \pi$:





First no-reflection mode:
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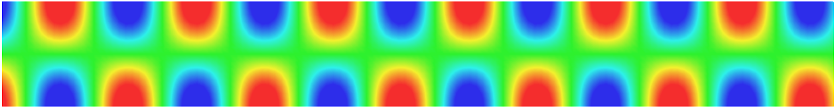
Second no-reflection mode:
$k = 2.8896\cdots$

There is a perfect agreement!

Below we represent $\Re e(u(x,y)e^{-i\omega t})$ with u...

...a no-reflection mode:

with the corresponding incident propagating mode:

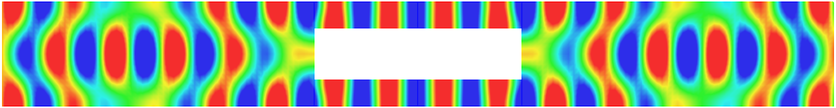


We observe no reflection but a phase shift in the transmitted wave.

No-reflection mode in the time-domain

Below we represent $\Re e(u(x,y)e^{-i\omega t})$ with u...

...a no-reflection mode:



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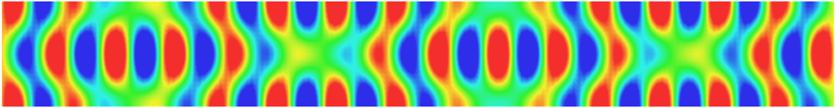

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$\mathcal{PT}$-SYMMETRY (SPACE-TIME REFLECTION SYMMETRY)

Remember that:

$$A_{\tilde{\alpha}} u = -\frac{1}{1+\rho(x,y)} \left(\tilde{\alpha}(x) \frac{\partial}{\partial x} \left(\tilde{\alpha}(x) \frac{\partial u}{\partial x} \right) + \frac{\partial^2 u}{\partial y^2} \right)$$

and that

$$\tilde{\alpha}(-x) = \overline{\tilde{\alpha}(x)}$$

$\mathcal{PT}$-symmetry (Space-time reflection symmetry)

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For a symmetric obstacle (i.e. $\rho(-x,y) = \rho(x,y)$), we have

$$A_{\tilde{\alpha}} \mathcal{Q} = \mathcal{Q} A_{\tilde{\alpha}}$$

where the operator $\mathcal{Q}$ is defined by $\mathcal{Q}u(x,y) = \overline{u(-x,y)}$

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We say that $A_{\tilde{\alpha}}$ is $\mathcal{PT}$-symmetric because $\mathcal{Q} = \mathcal{PT}$ where

$$\mathcal{P}u(x,y) = u(-x,y) \text{ and } \mathcal{T}u(x,y) = \overline{u(x,y)}$$

$\mathcal{P}$ stands for parity and $\mathcal{T}$ for "time reversal"

$\mathcal{PT}$-SYMMETRY (SPACE-TIME REFLECTION SYMMETRY)

SUMMARY

If the obstacle is symmetric:

$$A_{\tilde{\alpha}} \mathcal{Q} = \mathcal{Q} A_{\tilde{\alpha}}$$

where $\mathcal{Q} = \mathcal{PT}$ is such that

$$\begin{cases} \mathcal{Q}(\lambda u) = \overline{\lambda} \mathcal{Q} u \\ \mathcal{Q}^2 = I \end{cases}$$

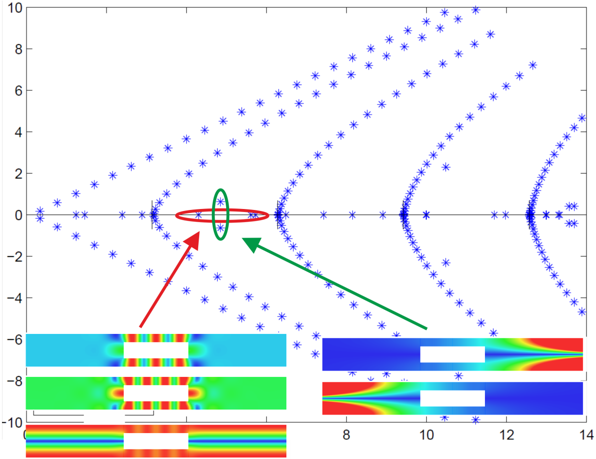
CONSEQUENCES

- the spectrum of $A_{\tilde{\alpha}}$ is stable by complex conjugation:
  $$\sigma(A_{\tilde{\alpha}}) = \overline{\sigma(A_{\tilde{\alpha}})}$$
- if $\lambda \in \mathbb{R}$ is a simple eigenvalue, then for the eigenfield u:
  $$|u(x, y)| = |u(-x, y)|$$

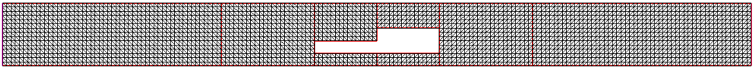
Modulus of eigenfields

By $\mathcal{PT}$-symmetry, if $\lambda \in \mathbb{R}$ is a simple eigenvalue, then:

$$|u(x,y)| = |u(-x,y)|$$



Here the scatterer is a not symmetric in x, and neither in y:



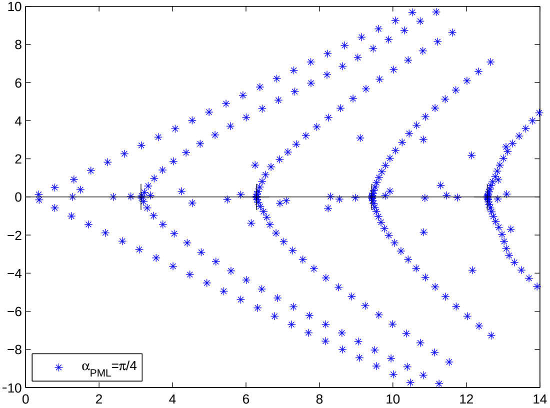
We expect:

- No trapped modes
- No invariance of the spectrum by complex conjugation

Numerical illustration

in a non $\mathcal{PT}$-symmetric case

Square root of the spectrum

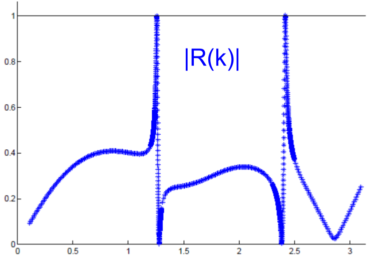


- The spectrum is no longer symmetric w.r.t. the real axis.
- There are several eigenvalues near the real axis.

Numerical illustration

in a non $\mathcal{PT}$-symmetric case

Again results can be validated by computing $R(k)$ for $0 < k < \pi$:



$k = 1.2803 + 0.0003i$

$k = 2.3868 + 0.0004i$

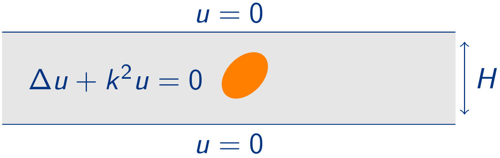
$k = 2.8650 + 0.0241i$

Complex eigenvalues also contain useful information about almost no-reflection.

Outline

Dirichlet waveguides

The same method applies for Dirichlet boundary conditions.



The main difference is the presence of the cut-off value $k_*^2 = \dfrac{\pi^2}{H^2}$.

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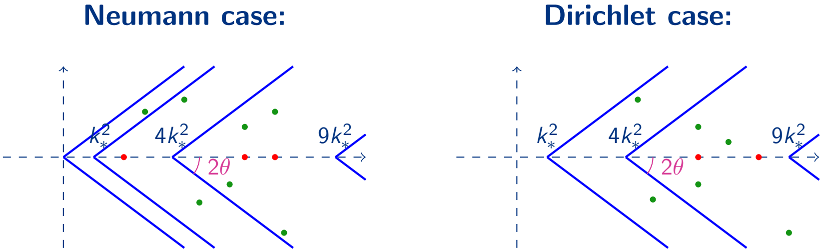
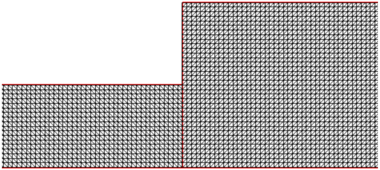
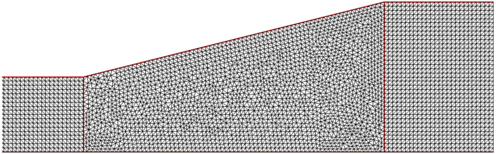


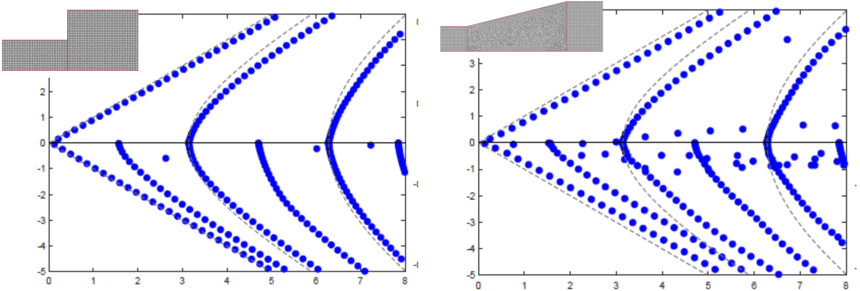
The main difference is the presence of the cut-off value $k_*^2 = \dfrac{\pi^2}{H^2}$.

The same method can be applied to the junction of two different waveguides.

Let us compare an abrupt junction with an "adiabatic" one :





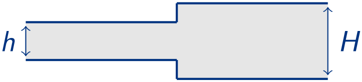


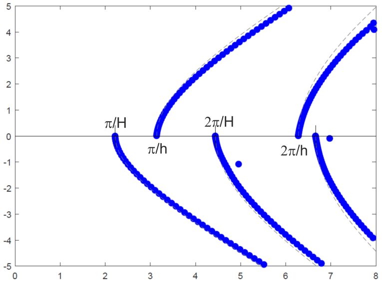
As expected:

- the essential spectrum is no-longer symetric;
- there are much more eigenvalues close to the real axis for the "adiabatic" junction.

Our approach can provide a tool to quantify the efficiency of the junction.

An interesting configuration is the junction of 2 different Dirichlet waveguides.

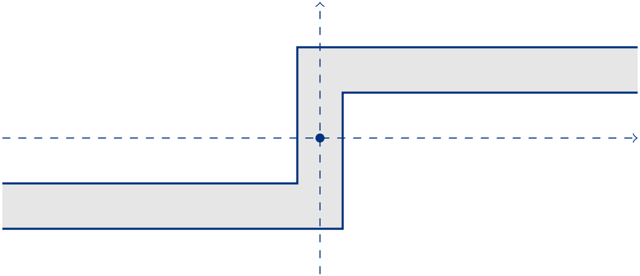




CONSEQUENCES

- Now $\mathbb{C}\backslash\sigma_{ess}(A_{\tilde{\alpha}})$ is a connected set!
- Our "new" eigenvalues correspond in fact to classical complex resonances in non-classical sheets of the Riemannn surface......

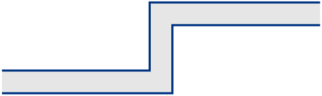
A PT-symmetric junction

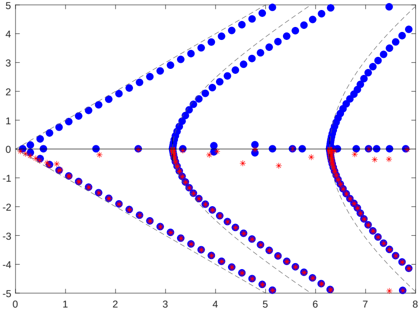


A new choice of Parity

Here $\mathcal{P}u(x,y) = u(-x,-y)$

A PT-symmetric junction

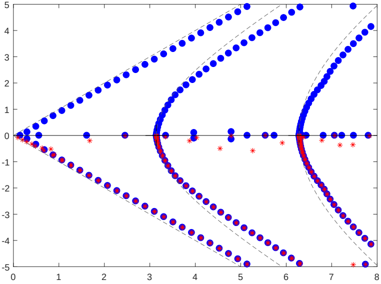


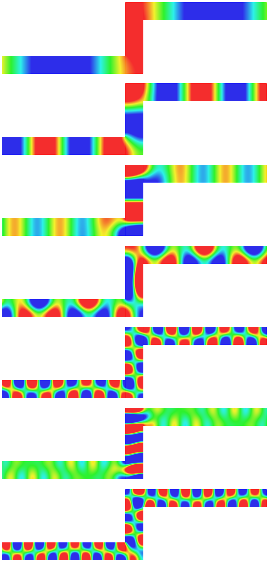


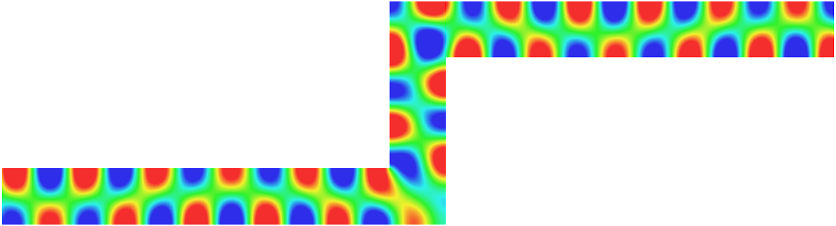
In red: classical complex scaling
In blue: conjugate complex scaling

We can check that there are no trapped modes (no red eigenvalues on the real axis).

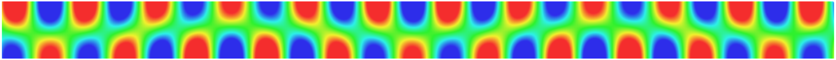
The modes associated to the 7 first real eigenvalues :

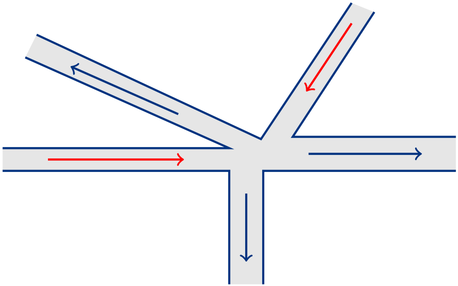






with the corresponding incident wave (which is a linear combination of 2 propagating modes):



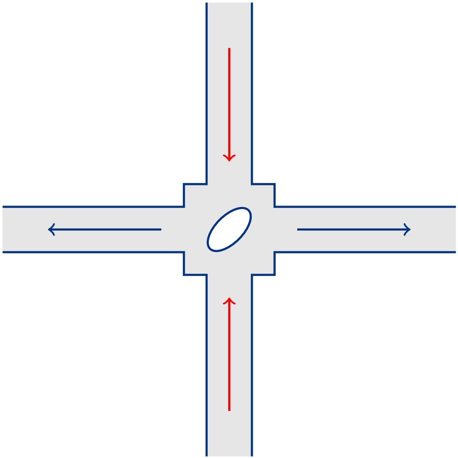


OBJECTIVE

Find (k, u) such that u is ingoing in some ports and outgoing in the others.

For an N-ports junction, there are 2^{N-1} such problems and corresponding spectra.

This is a bar-bar example of such problem:



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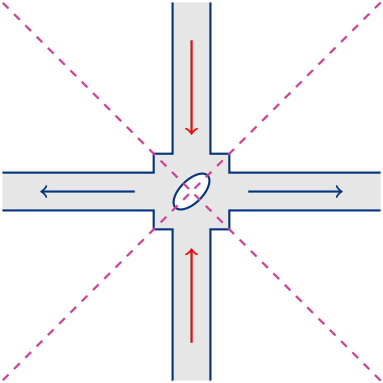
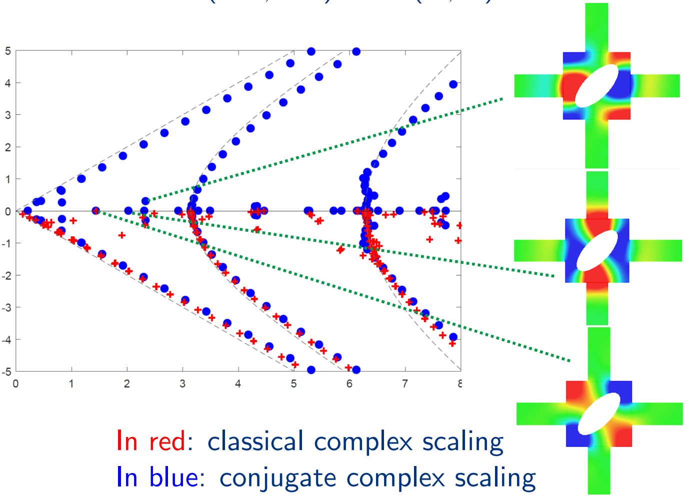


- There are two axes of $\mathcal{PT}$-symmetry!
- There is also a (classical) central symmetry.

The eigenmodes are all symmetric or antisymmetric:

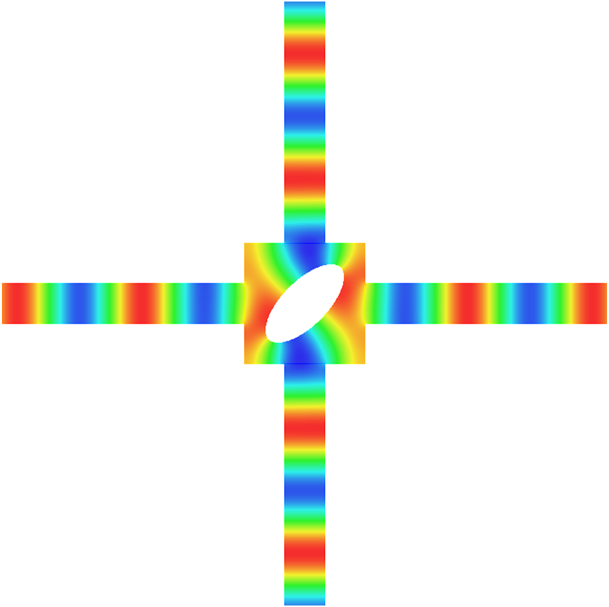
$$u(-X,-Y) = \pm u(X,Y)$$



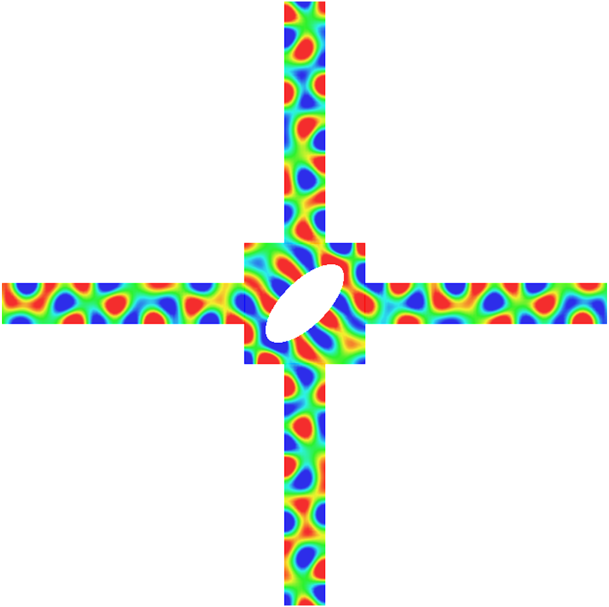
In red: classical complex scaling

In blue: conjugate complex scaling

Multiport waveguides junction

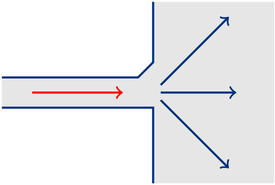


Multiport waveguides junction

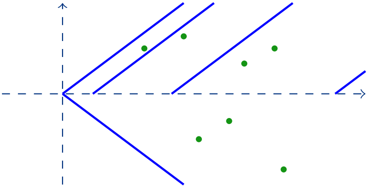


The baffled waveguide

A last (important) application concerns the radiation from a semi-infinite baffled waveguide:

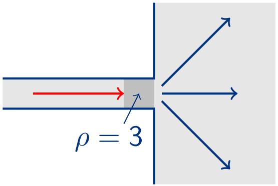


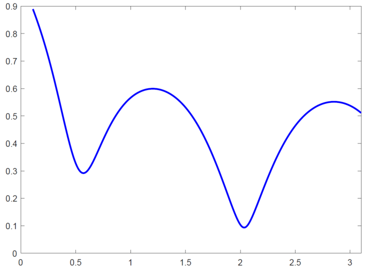
The expected spectrum is as follows:



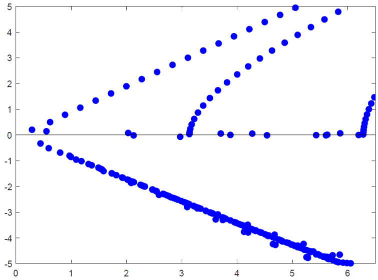
In the half-space, we apply a complex scaling in the radial cooordinate (radial PML).

The geometry:





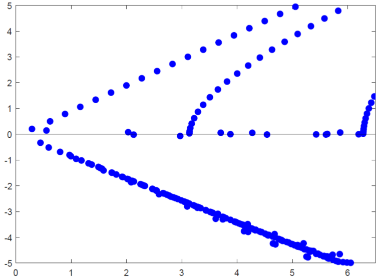
$|R(k)|$

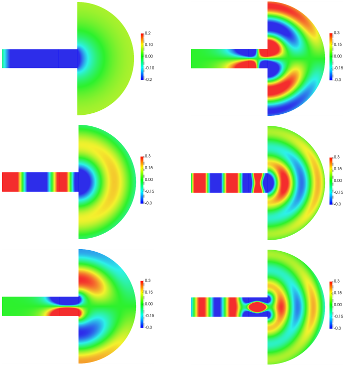


Again, minima of $|R(k)|$ correspond to eigenvalues near the real axis !

The baffled waveguide

The modes associated to the 6 first eigenvalues near the real axis:





Conclusion

There is still a lot of work to do !

- Treat the case of diffractive gratings.
- Justify the numerics (absence of spectral pollution).
- Clarify the link between our new spectrum and classical resonance frequencies.
- Find similar spectral approaches for other phenomena in waveguides (perfect invisibility, total reflection, modal conversion, etc...)
- ...

A part of these results have been published in:

Trapped modes and reflectionless modes as eigenfunctions of the same spectral problem, Anne-Sophie Bonnet-BenDhia, Lucas Chesnel and Vincent Pagneux, Proceedings of the Royal Society A, 2018.


Thank You