Absence of the singular spectrum in a twisted Dirichlet-Neumann waveguide

Ph. Briet[1] J.Dittrich[2] D. Krejčiřík[3]

[1]Centre de Physique Théorique,CNRS, Marseille-France
[2]Nuclear Physics Institute ASCR, Řež, Czech Republic
[3]Czech Technical University in Prague, Czech Republic

Differential Operators on Graphs and Waveguides
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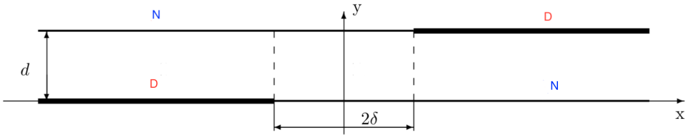
Outline

The Dittrich-Kříž twisted problem

Consider the following straight domain Ω in $\mathbb{R}^2$:

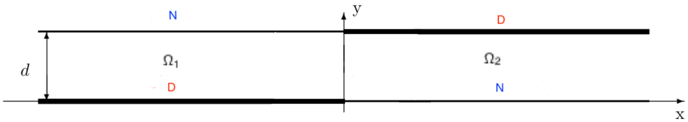


In this talk we consider the case $\delta = 0$

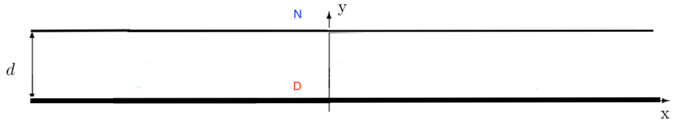
Consider the Laplace operator defined on $\mathcal{H} = L^2(\Omega)$ with DBC on *D* and NBC on *N*.

The Dittrich-Kříž twisted problem

Twisted system



free system



Main problem $\rightarrow$ study of the scattering theory

The Hamiltonian

Let

$$D(h) = \{\psi \in \mathcal{H}^1(\Omega) \mid \psi\lceil_D = 0\} \text{ and}$$

$$h[\psi] := \int_\Omega |\nabla\psi|^2 dx$$

Then

$$D(H) = \{\psi \in \mathcal{H}^1(\Omega), \Delta\psi \in \mathcal{H} \mid \psi\lceil_D = 0,\ \partial_y\psi\lceil_N = 0\}$$

$$H\psi = -\Delta\psi$$

The Hamiltonian

In fact we can prove that

If $\psi \in D(H)$ then $\psi \in \mathcal{H}^2(\Omega_0)$ for every open set $\Omega_0 \subset \Omega$ such that

$$\{(0,0),(0,d)\} \cap \overline{\Omega_0} = \emptyset$$

References

- J.Dittrich, J.Kříž, Jour. Math. Phys 2002
- Ph Briet, J. Dittrich, E. Soccorsi, Jour. Math Phys, 2014
- D. Krejčiřík, E. Zuazua Jour. Diff. Equat. 2011
- D. Krejčiřík, H. Kovarik Math Nachr. 2008
- ...

Questions

- Point spectrum
- Absence of singular continuous spectrum
- Completeness of the wave operators

Wave operators

Let $\Omega_1 = (0,d) \times \mathbb{R}^-$ and $\Omega_2 = (0,d) \times \mathbb{R}^+$ and χ_j the characteristic function of Ω_j.

Let $H_1 = -\Delta$ on Ω with BDC on $\{0\} \times \mathbb{R}$ and NBC on $\{d\} \times \mathbb{R}$,

$H_2 = -\Delta$ on Ω with BDC on $\{d\} \times \mathbb{R}$ and NBC on $\{0\} \times \mathbb{R}$

Prove the existence of the TWO wave operators : $j = 1, 2$

$$\Omega_j^{\mp} = s - \lim_{t \to \pm\infty} e^{itH} \chi_j e^{itH_j}$$

and

$$W_j^{\pm} = s - \lim_{t \to \pm\infty} e^{itH_j} \chi_j e^{itH} P_{ac}(H)$$

Here $P_{ac}(H_j) = \mathbb{I}_{\mathcal{H}}$

Completeness

Prove that

$$(\Omega_j^{\pm})^* = W_j^{\pm}$$

and the completeness relation

$$P_{ac}(H) = \Omega_1^{\pm} W_1^{\pm} + \Omega_2^{\pm} W_2^{\pm}$$

If $P_{ac}(H) = \mathbb{I}_{\mathcal{H}}$ i.e. $\sigma_{sing} = \emptyset \rightarrow$ asymptotic completeness

The spectrum

Theorem:

$$\sigma_{ess}(H) = [\mathcal{E}, +\infty);\ \mathcal{E} = \frac{\pi^2}{4d^2}$$

Proof : $\rightarrow$ J.Dittrich, J.Kříž

(See also Ph. Briet, H. Abdou Soimadou, D. Krejčiřík, to appear in ZAMP)

The spectrum

Denote by $\sigma_{pp}(H)$ the set of all eigenvalues of H and $\sigma_d(H)$ the set of discrete eigenvalues of H.

We know from J.Dittrich, J.Kříž that if $\delta > 0$ then $\sigma_d(H) \neq \emptyset$. We show that

Theorem: If $\delta = 0$ then $\sigma_{pp}(H) = \emptyset$

strategy of proof

Suppose that there exist an eigenvalue $\lambda \in \mathbb{R}$ and $\psi \in \mathcal{H}$ s.t.

$$H\psi = \lambda\psi$$

Then we construct a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $D(h)$ s.t.

$$2\|\partial_x \psi\|^2 = \lim_{n\to\infty} \big(h(\psi, \varphi_n) - \lambda(\psi, \varphi_n)\big) = 0$$

then regularity properties of ψ and the fact $\psi \in \mathcal{H}^1 \Rightarrow \psi = 0$.

In fact $\varphi_n = (\psi + 2x\partial_x\psi)\chi_n$ where
χ_n is an approximation of the identity function

Mourre estimate

Let $\mathcal{T} := \{E_k\}_{k\in\mathbb{N}^*}$ and let $E \in \mathbb{R} \setminus \mathcal{T}$ and $\eta > 0$, s.t.

$$(E-\eta, E+\eta) \cap \mathcal{T} = \emptyset, \quad P_\eta := P_{(E-\eta,E+\eta)}$$

The conjugate operator :

$$A = \frac{1}{2}(F(x)i\partial_x + i\partial_x F(x))$$

$F \in C^2(\mathcal{R})$, $F(x) \sim x$ in a neighbourhoud of $\pm\infty$

Theorem:

- There exists a positive number α and a compact operator K on such that

$$P_\eta i[H,A]P_\eta \geq \alpha P_\eta + P_\eta K P_\eta$$

It also holds

- $D(A) \cap D(H)$ is a core of H.
- e^{itA} leaves $D(H)$ invariant and $\sup_{|t|<1} \|e^{itA}\psi\| < \infty$, $\psi \in D(H)$,
- the form $i((H\psi, A\psi) - (A\psi, H\psi))$ on $D(A) \cap D(H)$ is bounded below. The associate operator B is s.t. $D(B) \supset D(H)$
- The operator associated to $i((B\psi, A\psi) - (A\psi, B\psi))$, is bounded from $D(H)$ to $D(H)^*$

See Georgescu-Gerard, JFA (2004) for a details about these conditions.

Mourre estimate

Corollary:

$$\sigma_{sc}(H) = \emptyset$$

So H is purely absolutely continuous and the asymptotic completeness holds.

elements of proof

Let $E \in (E_1, E_2), \eta$ as above and P_η the spectral projector of H_1. First we consider the operator H_1 (H_2), then choose A as the generator of dilation group,

$$A = \frac{1}{2}(xi\partial_x + i\partial_x x)$$

So a simple calculation shows that in a suitable sense

$$P_\eta i[H_1, A]P_\eta = -2P_\eta \partial_x^2 P_\eta = 2EP_\eta + 2(H_1 - E)P_\eta + 2P_\eta \partial_y^2 P_\eta$$

$$\geq 2\big(E - E_1 + o(\eta)\big)P_\eta$$

$\rightarrow$ a strict Mourre estimate for H_1 (H_2).

elements of proof

For the twisting model let $F \in C^\infty(\mathbb{R})$ st. $F(x) = x$ if $|x| > 1$ and $F(x) = 0$ elsewhere, in particular near $\{(0,0),(0,d)\}$.
Let

$$A = \frac{1}{2}(F(x)i\partial_x + i\partial_x F(x))$$

So

$$P_\eta[H,A]P_\eta = -P_\eta(F'\partial_x^2 + \partial_x^2 F')P_\eta = 2EP_\eta + P_\eta\big(F'(H_1 - E) + (H_1 - E)F'$$

$$+2P_\eta F'\partial_y^2 P_\eta + P_\eta 2E(F' - 1) + \frac{1}{2}F'''P_\eta$$

$$\geq 2\big(E - E_1 + o(\eta)\big)P_\eta + P_\eta K P_\eta$$

for some compact operator $K \to$ a Mourre estimate for H.

Thanks for your attention