Homogenization and uniform resolvent convergence for elliptic operators in a strip perforated along a curve

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Formulation of the problem

We consider an infinite planar straight strip

$\Omega := \{x : 0 < x_2 < d\}, \ d > 0$

perforated by small holes located closely one to another

along an infinite, or finite and closed, curve.

In Ω we consider a general second order elliptic operator subject to classical boundary conditions on the holes.

If the perforation is non-periodic and satisfies rather weak assumptions, we describe possible homogenized problems

The curve γ

Let γ be a curve

- lying in Ω and separated from $\partial\Omega$ by a fixed distance,

- is C^3-smooth, has no self-intersection,

- is either an infinite or finite closed curve.

Let us denote by:

- s its arc length, $s \in (-s^*, +s^*)$, where s^* is either finite or infinite

- $\rho = \rho(s) \in C^3(-s^*, +s^*)$ the vector function describing the curve γ.

The holes

Let us denote by:

- ε be a small positive parameter, $\mathbb{M}^\varepsilon \subset \mathbb{Z}$,

- for $k \in \mathbb{M}^\varepsilon$, $s_k^\varepsilon \in [-s^*, +s^*]$ set of points satisfying $s_k^\varepsilon < s_{k+1}^\varepsilon$.

- ω_k, $k \in \mathbb{Z}$, sequence of bounded domains in $\mathbb{R}^2$ having C^2-boundaries.

- the domain θ^ε defined by

$$\theta^\varepsilon := \theta_0^\varepsilon \cup \theta_1^\varepsilon, \quad \theta_i^\varepsilon := \bigcup_{k\in\mathbb{M}_i^\varepsilon} \omega_k^\varepsilon, \quad i = 0, 1,$$

$$\omega_k^\varepsilon := \{x : \varepsilon^{-1}\eta^{-1}(\varepsilon)(x - y_k^\varepsilon) \in \omega_k\}, \quad y_k^\varepsilon := \rho(s_k^\varepsilon),$$

where $\mathbb{M}_i^\varepsilon \subset \mathbb{Z}$, $\mathbb{M}_0^\varepsilon \cap \mathbb{M}_1^\varepsilon = \emptyset$, $\mathbb{M}_0^\varepsilon \cup \mathbb{M}_1^\varepsilon = \mathbb{Z}$,

- $\eta = \eta(\varepsilon)$ is a some function and $0 < \eta(\varepsilon) \leqslant 1$.

$\Omega^\varepsilon := \Omega \setminus \theta^\varepsilon$ perforated domain

(a) Perforation along an infinite curve (b) Perforation along a closed curve

Figure: Perforated domain

Remarks:

- The sizes of the holes and the distance between them are described by means of two small parameters.
- The perforation is quite general and no periodicity is assumed:both the shapes and the distribution of the holes can be rather arbitrary.

Formulation of the problem

$\mathcal{H}^\varepsilon$ singularly perturbed operator:

$$-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}A_{ij}\frac{\partial}{\partial x_j}+\sum_{j=1}^{2}A_j\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_j}\overline{A_j}+A_0 \qquad (1)$$

in Ω^ε subject to:

- Dirichlet condition on $\partial\Omega\cup\partial\theta_0^\varepsilon$
- Robin condition

$$\left(\frac{\partial}{\partial N^\varepsilon}+a\right)u=0 \quad \text{on} \quad \partial\theta_1^\varepsilon, \qquad \frac{\partial}{\partial N^\varepsilon}:=\sum_{i,j=1}^{2}A_{ij}\nu_i^\varepsilon\frac{\partial}{\partial x_j}+\sum_{j=1}^{2}\overline{A}_j\nu_j^\varepsilon,$$

where

- $\nu^\varepsilon=(\nu_1^\varepsilon,\nu_2^\varepsilon)$ is the inward normal to $\partial\theta_1^\varepsilon$,

- $a\in W_\infty^1(\{x:|\tau|<\tau_0\})$.

Formulation of the problem

- (s, τ) are the local coordinates introduced in the vicinity of γ,

- τ is the distance to a point measured along the normal ν^0 to γ which is inward for Ω_-

- Ω_- and Ω_+ are the partitions of Ω originated by γ.

Remarks:

- On the boundary of the holes we impose Dirichlet or Neumann or Robin condition.
- Boundaries of different holes can be subject to different types of boundary conditions.
- Such mixtures of boundary conditions were not considered before.

Physical interpretation

Waveguide theory:

- Our operator describes a quantum particle in a waveguide

- The waveguide is not isotropic: coefficients of the operator variable.

- The perforation represents small defects distributed along a given line,

- Conditions on the boundaries of the holes impose certain regime:

Dirichlet condition describes a wall and the particle can not pass through such boundary.

- The homogenization describes the effective behavior of our model once the perforation becomes finer.

- The type of resolvent convergence characterizes in which sense the perturbed model is close to the effective one

Operator Ω^ε

- Sesquilinear form

$$\begin{aligned}\mathfrak{a}^\varepsilon(u,v) := \sum_{i,j=1}^{2}\left(A_{ij}\frac{\partial u}{\partial x_j},\frac{\partial v}{\partial x_i}\right)_{L_2(\Omega^\varepsilon)} + \sum_{j=1}^{2}\left(A_j\frac{\partial u}{\partial x_j},v\right)_{L_2(\Omega^\varepsilon)} \\ + \sum_{j=1}^{2}\left(u, A_j\frac{\partial v}{\partial x_j}\right)_{L_2(\Omega^\varepsilon)} + (A_0 u, v)_{L_2(\Omega^\varepsilon)}\end{aligned} \tag{2}$$

in $L_2(\Omega^\varepsilon)$ on the domain $W_2^1(\Omega^\varepsilon)$.

- $\mathcal{H}^\varepsilon$ self-adjoint operator in $L_2(\Omega^\varepsilon)$

associated with the sesquilinear form

$$\mathfrak{h}^\varepsilon(u,v) := \mathfrak{a}^\varepsilon(u,v) + (au,v)_{L_2(\partial\theta_1^\varepsilon)}$$

in $L_2(\Omega^\varepsilon)$ on $\mathring{W}_2^1(\Omega^\varepsilon, \partial\Omega \cup \partial\theta_0^\varepsilon)$.

- $\mathring{W}_2^1(\Omega^\varepsilon, \partial\Omega \cup \partial\theta_0^\varepsilon)$ functions in $W_2^1(\Omega^\varepsilon)$ with zero trace on $\partial\Omega \cup \partial\theta_0^\varepsilon$.

Formulation of the problem

Aim:

to study the resolvent convergence and the spectrum's behavior of the operator

$$\mathcal{H}^\varepsilon \text{ as } \varepsilon \to +0,$$

i.e. the asymptotic behavior of the resolvent of such operator as ε *tends to zero*

Formulation of the problem

- Effective operator $\mathcal{H}_{\mathrm{D}}^{0}$:

operator (1) in $L_2(\Omega)$ subject to the Dirichlet condition on γ and $\partial\Omega$.

- Associated form: $\mathfrak{h}_{\mathrm{D}}^{0}(u,v) := \mathfrak{a}(u,v)$ in $L_2(\Omega)$ on $\mathring{W}_2^1(\Omega, \partial\Omega \cup \gamma)$,

- $\mathfrak{a}$: form (2), where Ω^{ε} is replaced by Ω.

- Domain of operator $\mathcal{H}_{\mathrm{D}}^{0}$: $\mathfrak{D}(\mathcal{H}_{\mathrm{D}}^{0}) = \mathring{W}_2^1(\Omega, \partial\Omega \cup \gamma) \cap W_2^2(\Omega \setminus \gamma)$.

Assumptions on holes

A1 There exist $0 < R_1 < R_2$, $b > 1$, $L > 0$ and $x^k \in \omega_k$, $k \in \mathbb{M}^\varepsilon$, such that

$$B_{R_1}(x^k) \subset \omega_k \subset B_{R_2}(0), \quad |\partial\omega_k| \leqslant L \quad \text{for each} \quad k \in \mathbb{M}^\varepsilon,$$

$$B_{bR_2\varepsilon}(y_k^\varepsilon) \cap B_{bR_2\varepsilon}(y_i^\varepsilon) = \emptyset \quad \text{for each} \quad i, k \in \mathbb{M}^\varepsilon, \quad i \neq k,$$

and for all sufficiently small ε.

Assumptions on holes

Remarks:

- the sizes of holes are of the same order and there is a minimal distance between them.
- no periodicity for the perforation is assumed.
- since $\mathbb{M}^\varepsilon$ is arbitrary, number of holes can be infinite or finite
- in the latter case, by an appropriate choice of $\mathbb{M}^\varepsilon$, the distances between the holes can be even not small, but finite.

Assumptions on holes

(A2) For b and R_2 in (A1) and $k \in \mathbb{M}^\varepsilon$ there exists a generalized solution

$X_k : B_{b_* R_2}(0) \setminus \omega_k \mapsto \mathbb{R}^2$, $b_* := (b+1)/2$, of

$$\begin{aligned} \operatorname{div} X_k = 0 \quad &\text{in} \quad B_{b_* R_2}(0) \setminus \omega_k, \\ X_k \cdot \nu = -1 \quad &\text{on} \quad \partial\omega_k, \\ X_k \cdot \nu = \varphi_k \quad &\text{on} \quad \partial B_{b_* R_2}(0), \end{aligned} \tag{3}$$

- belonging to $L_\infty(B_{b_* R_2}(0) \setminus \omega_k)$

- bounded uniformly in $k \in \mathbb{M}^\varepsilon$ in $L_\infty(B_{b_* R_2}(0) \setminus \omega_k)$.

ν is the outward normal to $\partial B_{b_* R_2}(0)$ and to $\partial\omega_k$

$\varphi_k \in L_\infty(\partial B_{b_* R_2}(0))$ satisfying

$$\int_{\partial B_{b_* R_2}(0)} \varphi_k \, ds = |\partial\omega_k|. \tag{4}$$

Assumptions on holes

Remarks:

Assumption (A2) is a restriction for the geometry of boundaries $\partial\omega_k$.

- Problem (3) can be rewritten to the Neumann problem for the Laplace equation by letting $X_k = \nabla V_k$.
- Then identity (4) is the solvability condition and this is the only restriction for φ_k we suppose. Problem (3) is solvable for each fixed k

  and its solution belongs to $L_\infty(B_{b_* R_2}(0) \setminus \omega_k)$.
- we assume that the norm $\|X_k\|_{L_\infty(B_{b_* R_2}(0)\setminus\omega_k)}$ is bounded uniformly in k.

Main Result

Theorem

Let us assume

$$\varepsilon \ln \eta(\varepsilon) \to 0, \quad \varepsilon \to +0, \tag{5}$$

suppose (A1), (A2), and

- (A3) *There exists a constant* $R_3 > bR_2$ *such that*

$$\{x : |\tau| < \varepsilon b R_2\} \subset \bigcup_{k \in \mathbb{M}_0^\varepsilon} B_{R_3\varepsilon}(y_k^\varepsilon), \quad \omega_k^\varepsilon \subset B_{R_3\varepsilon}(y_k^\varepsilon) \quad \forall k \in \mathbb{M}_0^\varepsilon.$$

Then the estimate

$$\|(\mathcal{H}^\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}_{\mathrm{D}}^0 - \mathrm{i})^{-1}\|_{L_2(\Omega) \to W_2^1(\Omega^\varepsilon)} \leqslant C\varepsilon^{\frac{1}{2}} \left(|\ln \eta(\varepsilon)|^{\frac{1}{2}} + 1 \right) \tag{6}$$

holds true, where C *is a positive constant independent of* ε.

Main Result

Assumptions:

- (5): the sizes of the holes are not too small
- (A3): the holes with the Dirichlet condition are, roughly speaking, distributed “uniformly”

Results:

- homogenized operator is subject to the Dirichlet condition on γ
- norm resolvent condition in the sense of the operator norm $\| \cdot \|_{L_2(\Omega)\to W_2^1(\Omega^\varepsilon)}$

Main Result

Remark:

Relation (5) admits the situation when the sizes of the holes are much smaller than the distances between them

for example, $\eta(\varepsilon) = \varepsilon^{\alpha}$, $\alpha = \mathit{const} > 0$,

nevertheless the homogenized operator is still subject to the Dirichlet condition on γ.

Main Result

This phenomenon is close to a similar one for the operators with frequent alternation of boundary conditions,

- Borisov, D., Cardone, G.: Homogenization of the planar waveguide with frequently alternating boundary conditions. J. Phys. A. 42, id 365205 (2009)
- D. Borisov, R. Bunoiu, G. Cardone, Waveguide with non-periodically alternating Dirichlet and Robin conditions: homogenization and asymptotics. ZAMP 64 (2013), 439-472.
- Chechkin, G.A.: Averaging of boundary value problems with singular perturbation of the boundary conditions. Russ. Acad. Sci. Sb. Math. 79, 191-220 (1994)

Main Result

$\mathcal{H}^0_\beta$: operator (1) subject to the boundary conditions

$$[u]_\gamma = 0, \quad \left[\frac{\partial u}{\partial N^0}\right]_\gamma + \beta u\Big|_\gamma = 0. \tag{7}$$

- $[\cdot]_\gamma$ denote the jump of a function on γ, i.e. $[v]_\gamma = v\big|_{\tau=+0} - v\big|_{\tau=-0}$.

$$\frac{\partial}{\partial N^0} := \sum_{i,j=1}^{2} A_{ij}\nu_i^0 \frac{\partial}{\partial x_j}. \qquad \nu^0 = (\nu_1^0, \nu_2^0)$$

- $\beta = \beta(s) \in W^1_\infty(\gamma)$.

Boundary condition (7) describes a delta-interaction on γ,

Albeverio, S., Gesztesy, F., Hegh-Krohn, R., Holden, H.: Solvable models in quantum mechanics. AMS Chelsea (2005)

Main Result

The associated form is

$$\mathfrak{h}_\beta^0(u,v) := \mathfrak{a}(u,v) + (\beta u, v)_{L_2(\gamma)} \text{ in } L_2(\Omega) \text{ in } \mathring{W}_2^1(\Omega).$$

One can show that

$$\mathfrak{D}(\mathcal{H}_\beta^0) = \{u \in \mathring{W}_2^1(\Omega) : u \in W_2^2(\Omega_\pm) \text{ and (7) is satisfied}\}.$$

If $\beta = 0$, $\mathcal{H}_0^0 = \mathcal{H}^0$.

In this case there is no boundary condition on γ and

the domain of $\mathcal{H}^0$ is $\mathfrak{D}(\mathcal{H}^0) = \mathring{W}_2^1(\Omega) \cap W_2^2(\Omega)$.

Second Main Result

Perturbed operator involves the Dirichlet condition at least on a part of $\partial\theta^\varepsilon$

$\varepsilon \ln \eta(\varepsilon)$ converges either to a non-zero constant or to infinity.

Theorem

Suppose: (A1), (A2),

$$\frac{1}{\varepsilon \ln \eta(\varepsilon)} \to -\rho, \quad \varepsilon \to +0, \tag{8}$$

- $\mathbb{M}_0^\varepsilon$ *be non-empty (there are holes with the Dirichlet condition)*

- *For b and R_2 in (A1) and $s \in \mathbb{R}$ we denote*

$$\alpha^\varepsilon(s) := \begin{cases} \dfrac{\pi}{bR_2}, & |s - s_k^\varepsilon| < bR_2\varepsilon, \quad k \in \mathbb{M}_0^\varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Second Main Result

A4 There exist

- $\alpha = \alpha(s) \in W^1_\infty(\gamma)$ and $\varkappa(\varepsilon) \to +0$, $\varepsilon \to +0$,

such that for all sufficiently small ε the estimate

$$\sum_{q\in\mathbb{Z}} \frac{1}{|q|+1} \left| \int_n^{n+\ell} (\alpha^\varepsilon(s) - \alpha(s)) \mathrm{e}^{-\frac{\mathrm{i}q}{2\pi\ell}(s-n)}\, ds \right|^2 \leqslant \varkappa^2(\varepsilon) \qquad (9)$$

is valid, where - $n = -s_*$, $\ell = 2s_*$, if γ is a finite curve, and

- $n \in \mathbb{Z}$, $\ell = 1$, if γ is an infinite curve (in this case estimate (9) is supposed to hold uniformly in n).

The sum in the left hand side of (9) is nothing but the norm in $W_2^{-\frac{1}{2}}(0, \ell)$.

Second Main Result

Then the estimates hold

$$\|(\mathcal{H}^\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}^0_\beta - \mathrm{i})^{-1}\|_{L_2(\Omega)\to L_2(\Omega^\varepsilon)} \leqslant C\big(\varepsilon^{\frac{1}{2}} + \varkappa(\varepsilon)\big) \tag{10}$$

$$\|(\mathcal{H}^\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}^0_{\beta_0} - \mathrm{i})^{-1}\|_{L_2(\Omega)\to L_2(\Omega^\varepsilon)} \leqslant C\big(\varepsilon^{\frac{1}{2}} + \varkappa(\varepsilon) + \mu(\varepsilon)\big) \tag{11}$$

where

$$\beta := \alpha\frac{(\rho+\mu)}{A_{11}A_{22} - A_{12}^2}, \quad \beta_0 := \alpha\frac{\rho}{A_{11}A_{22} - A_{12}^2}, \quad \mu(\varepsilon) := -\frac{1}{\varepsilon \ln \eta(\varepsilon)} - \rho.$$

Results:

- homogenized operator has boundary condition (7) on γ
- norm resolvent convergence holds in the sense of the operator norm $\|\cdot\|_{L_2(\Omega)\to L_2(\Omega^\varepsilon)}$ only.

Remark:

Similar situation holds for the problems with frequent alternation of boundary conditions

with the Dirichlet conditions on exponentially small parts of the boundary

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- Borisov, D., Bunoiu, R., G.C.: Compt. Rend. Math. **349**, 53-56 (2011)
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Second Main Result

Assumption (A4):

- β in boundary condition (7) for the homogenized operator

  depends only on the distribution of the points s_k^ε and

  there is no dependence on the geometries of the holes.

- There are also no special restrictions for part $\partial\theta_0^\varepsilon$ with the Dirichlet condition.

  For instance, the number of holes in $\partial\theta_0^\varepsilon$ can be finite or infinite and

  the distribution of this set can be very arbitrary.

Main Result

To improve the norm:

- employ the boundary corrector, see (12), or

- assume additionally $\rho = 0$, see (13).

There exists an explicit function W^ε such that the following estimate hold

$$\|(\mathcal{H}^\varepsilon - \mathrm{i})^{-1} - (1 - W^\varepsilon)(\mathcal{H}^0_\beta - \mathrm{i})^{-1}\|_{L_2(\Omega)\to W_2^1(\Omega^\varepsilon)} \leqslant C\big(\varepsilon^{\frac{1}{2}} + \varkappa(\varepsilon)(\rho + \mu(\varepsilon))\big) \tag{12}$$

If $\rho = 0$, the following estimate hold

$$\|(\mathcal{H}^\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}^0 - \mathrm{i})^{-1}\|_{L_2(\Omega)\to W_2^1(\Omega^\varepsilon)} \leqslant C\big(\varepsilon^{\frac{1}{2}} + \mu^{\frac{1}{2}}(\varepsilon)\big). \tag{13}$$

$\mathbb{M}_0^\varepsilon = \emptyset$: No Dirichlet condition on $\partial\theta^\varepsilon$, only Robin cond.

Theorem

Suppose: (A1), (A2),

- $\mathbb{M}_0^\varepsilon$ *is empty*

- *either* $a \equiv 0$ *or* $\eta(\varepsilon) \to 0$, $\varepsilon \to +0$.

Then the following estimates hold:

- *if* $a \not\equiv 0$, $\eta \to +0$,

$$\|(\mathcal{H}^\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}^0 - \mathrm{i})^{-1}\|_{L_2(\Omega)\to W_2^1(\Omega^\varepsilon)} \leqslant C\eta(\varepsilon)|\ln \eta(\varepsilon)|^{\frac{1}{2}}, \tag{14}$$

- *if* $a \equiv 0$,

$$\|(\mathcal{H}^\varepsilon - \mathrm{i})^{-1}f - (\mathcal{H}^0 - \mathrm{i})^{-1}f\|_{L_2(\Omega)\to W_2^1(\Omega^\varepsilon)} \leqslant C\varepsilon^{\frac{1}{2}}\eta(\varepsilon)(|\ln \eta(\varepsilon)|^{\frac{1}{2}} + 1), \tag{15}$$

$\mathbb{M}_0^\varepsilon = \emptyset$

Results:

- homogenized operator has no condition on γ
- norm resolvent convergence in the operator norm $\|\cdot\|_{L_2(\Omega)\to W_2^1(\Omega^\varepsilon)}$.

Remarks:

- Since $\eta(\varepsilon) \to +0$ or $a \equiv 0$, we needed no additional restrictions on holes.
- But if η is constant, we have to introduce Assumption (A5) in next theorem.

$\mathbb{M}_0^\varepsilon = \emptyset$

Theorem

Suppose: (A1), (A2),

- $\eta = \mathrm{const}$,

- $\mathbb{M}_0^\varepsilon$ *is empty.*

For b and R_2 in (A1) we denote

$$\alpha^\varepsilon(s) := \begin{cases} \dfrac{|\partial\omega_k|\eta}{2bR_2}, & |s - s_k^\varepsilon| < bR_2\varepsilon, \quad k \in \mathbb{M}^\varepsilon, \\ 0, & \textit{otherwise}. \end{cases}$$

$\mathbb{M}_0^\varepsilon = \emptyset$

Suppose also that

(A5) There exist

- $\alpha = \alpha(s) \in W^1_\infty(\gamma)$ and a function

- $\varkappa = \varkappa(\varepsilon)$, $\varkappa(\varepsilon) \to +0$, $\varepsilon \to +0$,

such that the estimates (9) hold.

Then the estimate hold

$$\|(\mathcal{H}^\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}^0_{\alpha \mathrm{a}} - \mathrm{i})^{-1}\|_{L_2(\Omega) \to W_2^1(\Omega^\varepsilon)} \leqslant C\big(\varepsilon^{\frac{1}{2}} + \varkappa(\varepsilon)\big). \qquad (16)$$

Results:

- homogenized operator has condition (7) on γ
- norm resolvent convergence in the operator norm $\|\cdot\|_{L_2(\Omega)\to W_2^1(\Omega^\varepsilon)}$.

Remarks:

- Assumption (A5): the lengths of $\partial\omega_k$ should be distributed rather smoothly to satisfy (9).
- coefficient β in (7) for the homogenized operator depends both on the distribution of the holes and the sizes of their boundaries.

Assumptions (A4) and (A5)

(A4) and (A5) are the same assumption but adapted for two different cases.

This estimate obviously holds true for a periodic perforation.

Example of a non-periodic perforation:

we start with a strictly periodic perforation along an infinite curve

but then we change the geometry and locations of a part of holes

so that the total number of deformed holes associated with each segment $s \in (q, q+1)$, $q \in \mathbb{Z}$,

is relatively small in comparison with unchanged holes.

Then inequality (9) is still true.

Assumptions (A4) and (A5)

Conjecture:

Assumptions (A4) and (A5) can not be improved or omitted to have a norm resolvent convergence.

In fact, they are employed only in following Lemma

Lemma

Function α_ε is bounded uniformly in ε in the norm of space $L_\infty(\gamma)$. The estimate

$$\left|\left((\alpha^\varepsilon - \alpha)a\widetilde{u}^0, v^\varepsilon\right)_{L_2(\widetilde{\gamma})}\right| \leqslant C\varkappa(\varepsilon)\|f\|_{L_2(\Omega)}\|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}$$

holds true.

and all the inequalities in the proof of this lemma are sharp.

Assumptions (A4) and (A5)

Way to simplify (9):

estimating $W_2^{-\frac{1}{2}}(0,\ell)$-norm by $L_2(0,\ell)$-norm.

Then (9) can be replaced by

$$\|\alpha^\varepsilon - \alpha\|^2_{L_2(n,n+\ell)} \leqslant C\|\alpha^\varepsilon - \alpha\|_{L_1(n,n+\ell)} \leqslant \varkappa^2(\varepsilon),$$

where we have employed the boundedness of α^ε, see previous Lemma.

However, this condition happens to be too restrictive and is satisfied just by few examples.

Convergence of the spectrum of $\mathcal{H}^\varepsilon$

Theorem

Under the hypotheses of Theorems 1–4,

the spectrum of perturbed operator $\mathcal{H}^\varepsilon$ converges to the spectrum of corresponding homogenized operator.

Namely,

- *if λ is not in the spectrum of the homogenized operator, for sufficiently small ε the same is true for the perturbed operator.*
- *if λ is in the spectrum of the homogenized operator, for each ε there exists λ_ε in the spectrum of the perturbed operator such that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to +0$.*

Convergence of the spectrum of $\mathcal{H}^{\varepsilon}$

Remarks:

Result on the spectrum is not implied immediately by previous Theorems.

In fact, even if they state the convergence of the perturbed resolvent to a homogenized one in the norm sense,

the norm is ε-dependent.

Convergence of the spectrum of $\mathcal{H}^\varepsilon$

Idea of proof

We consider the Laplacian subject to Dirichlet condition on all the holes.

The resolvent of its homogenized operator is the limit, in norm sense,

of the resolvent of the direct sum of the perturbed operator and of the considered Laplacian.

Now the norm is independent on ε and

the spectrum of the added operator in small holes tends to infinity in the sense that the bottom of this additional spectrum starts from $C\varepsilon^{-2}$.

Hence, the low part of the spectrum converges and this is what we need.

Main Results

- Description of the homogenized problems depending on the:

geometry,
sizes,
distribution of the holes
conditions on the boundary of the holes.

- homogenized operator has the same differential expression as the original operator,

but on the reference curve with

Dirichlet condition or
delta-interaction or
no condition.

- The norm resolvent convergence of the perturbed operator to the homogenized one.

Main Results

- The estimates for the rates of convergence.

- In general, the operator norm is from L_2 into W_2^1;

- in one case it is from L_2 into L_2,

    but it can be replaced by the norm from L_2 into W_2^1 employing a special boundary corrector .

Such kind of results on norm resolvent convergence are completely new for the domains perforated periodically along curves or manifolds,

especially because they hold for general non-periodic perforation with arbitrary boundary conditions.

Idea of Proofs

Our technique is based on the variational formulations of the equations for the perturbed and the homogenized operators.

We use no smoothing operator like previous papers on the operators with fast oscillating coefficients.

We write the integral identity for the difference of the perturbed and homogenized resolvents and

estimate then the terms coming from the boundary conditions.

It requires certain accurate estimates for various boundary integrals over holes and over the reference curve.

Idea of Proofs

The main difference of our technique is the assumptions for the perforation.

In previous works, (Belyaev, Chechkin, Gomez, Lobo, Oleinik, Perez, Shaposhnikova,...):

existence of an operator of continuation on the holes and uniform estimates for this operator.

We assume the solvability of a certain fixed boundary value problems for the divergence operator in a neighborhood of the holes.

We believe that our assumptions are not worse than the existence of the continuation operator since

we require just a solvability of certain boundary value problem

while the existence of the continuation operator means the possibility to extend *each* function in a given Sobolev space.

Homogenized Dirichlet condition: Theorem 1

If $f \in L_2(\Omega)$, we denote

$$u^\varepsilon := (\mathcal{H}^\varepsilon - \mathrm{i})^{-1} f, \qquad u^0 := (\mathcal{H}^0_{\mathrm{D}} - \mathrm{i})^{-1} f.$$

Estimate (6) is equivalent to

$$\|u^\varepsilon - u^0\|_{W_2^1(\Omega^\varepsilon)} \leqslant C\varepsilon^{\frac{1}{2}}\big(|\ln\eta|^{\frac{1}{2}} + 1\big)\|f\|_{L_2(\Omega)}, \tag{17}$$

Main idea:

employ the integral identities for u^ε and u^0 and get then a similar identity for $u^\varepsilon - u^0$.

But $u^\varepsilon - u^0$ does not satisfy Dirichlet condition on $\partial\theta_0^\varepsilon$ and

we can not use it as the test function in the integral identity for u^ε.

Homogenized Dirichlet condition

To overcome this difficulty, we make use of a boundary corrector: let

- $\chi_1(t) = 1$ if $t < 1$, $\qquad \chi_1(t) = 0$ if $t > 2$.

- $\chi_1^\varepsilon(x) := \chi_1\left(\frac{|\tau|}{R_3\varepsilon}\right)$ if $|\tau| < \tau_0$, $\qquad \chi_1^\varepsilon(x) := 0$ outside $\{x : |\tau| < \tau_0\}$,

- $v^\varepsilon := u^\varepsilon - u^0 + \chi_1^\varepsilon u^0 = u^\varepsilon - (1 - \chi_1^\varepsilon)u^0$.

So v^ε vanishes on $\partial\theta_0^\varepsilon$ and

we use it as test function in the integral identity for u^ε.

Our strategy: estimate independently $W_2^1(\Omega^\varepsilon)$-norm of v^ε and $\chi_1^\varepsilon u^0$.

This will lead us estimate (17).

Robin condition: Theorem 3

If $f \in L_2(\Omega)$, we denote

$$u^\varepsilon := (\mathcal{H}^\varepsilon - \mathrm{i})^{-1} f, \qquad u^0 := (\mathcal{H}^0 - \mathrm{i})^{-1} f, \qquad v^\varepsilon := u^\varepsilon - u^0.$$

Estimates (14) and (15) for the resolvents are equivalent to

$$\|u^\varepsilon - u^0\|_{W_2^1(\Omega^\varepsilon)} \leqslant C\eta(\varepsilon)\big(|\ln\eta|^{\frac{1}{2}} + 1\big)\|f\|_{L_2(\Omega)}, \quad a \not\equiv 0, \quad \eta \to +0, \tag{18}$$

$$\|u^\varepsilon - u^0\|_{W_2^1(\Omega^\varepsilon)} \leqslant C\varepsilon^{\frac{1}{2}}\eta(|\ln\eta|^{\frac{1}{2}} + 1)\|f\|_{L_2(\Omega)}, \quad a \equiv 0. \tag{19}$$

By the assumption $\mathbb{M}_0^\varepsilon = \emptyset$ we have $\theta_0^\varepsilon = \emptyset$, $\theta_1^\varepsilon = \theta^\varepsilon$.

Since $u^0 \in W_2^2(\Omega)$, $\left(\frac{\partial}{\partial N^\varepsilon} + a\right) u^0 \in L_2(\partial\theta^\varepsilon)$.

Robin condition: Theorem 3

Then v^ε is the generalized solution to the boundary value problem

$$\left(-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}A_{ij}\frac{\partial}{\partial x_j}+\sum_{j=1}^{2}A_j\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_j}\overline{A_j}+A_0-\mathrm{i}\right)v^\varepsilon=0\quad\text{in}\quad\Omega^\varepsilon,$$

$$v^\varepsilon=0\quad\text{on}\quad\partial\Omega,\qquad\left(\frac{\partial}{\partial N^\varepsilon}+a\right)v^\varepsilon=-\left(\frac{\partial}{\partial N^\varepsilon}+a\right)u^0\quad\text{on}\quad\partial\theta^\varepsilon.$$

Taking v^ε as the test function, we write the associated integral identity

$$\mathfrak{h}^\varepsilon(v^\varepsilon,v^\varepsilon)-\mathrm{i}\|v^\varepsilon\|^2_{L_2(\Omega^\varepsilon)}=-\left(\left(\frac{\partial}{\partial N^\varepsilon}+a\right)u^0,v^\varepsilon\right)_{L_2(\partial\theta^\varepsilon)}.\tag{20}$$

The main idea of our proof is to estimate the right hand side of this identity and

to get then the desired estimate for v^ε.

Robin condition: Theorem 4

If $f \in L_2(\Omega)$, let us denote

$$u^\varepsilon := (\mathcal{H}^\varepsilon - \mathrm{i})^{-1} f, \qquad u^0 := (\mathcal{H}^0_{\alpha\mathrm{a}} - \mathrm{i})^{-1} f.$$

We need to prove the estimate

$$\|u^\varepsilon - u^0\|_{W_2^1(\Omega^\varepsilon)} \leqslant C(\varepsilon^{\frac{1}{2}} + \varkappa)\|f\|_{L_2(\Omega)}. \tag{21}$$

In this case, curve γ can cross the holes while

the functions in the domain of homogenized operator $\mathcal{H}^0_{\alpha\mathrm{a}}$ have a jump of the normal derivative at this curve.

It causes troubles in getting integral identity for $u^\varepsilon - u^0$ and in further estimating.

Robin condition: Theorem 4

So we consider curve $\widetilde{\gamma} := \{x : \tau = -(b+1)R_2\varepsilon,\ s \in \mathbb{R}\}$ that does not intersect the holes by (A1) and

so we can get an estimate similar to (21) for $v^\varepsilon := u^\varepsilon - \widetilde{u}^0$,

where $\widetilde{u}^0 := (\widetilde{\mathcal{H}}^0_{\alpha\mathrm{a}} - \mathrm{i})^{-1} f$ and

$\widetilde{\mathcal{H}}^0_{\alpha\mathrm{a}}$ is the operator with the differential expression (1) subject to the boundary conditions

$$[u]_{\widetilde{\gamma}} = 0, \quad \left[\frac{\partial u}{\partial \widetilde{N}^0}\right]_{\widetilde{\gamma}} + (\alpha a)u\big|_{\widetilde{\gamma}} = 0, \tag{22}$$

$$\frac{\partial}{\partial \widetilde{N}^0} := \sum_{i,j=1}^{2} A_{ij}\nu_i^0 \frac{\partial}{\partial x_j}, \quad [u]_{\widetilde{\gamma}} := u\big|_{\tau=-(b+1)R_2\varepsilon+0} - u\big|_{\tau=-(b+1)R_2\varepsilon-0}.$$

Estimating $u^0 - \widetilde{u}^0$ by a previous Lemma, we get (21).

Homogenized delta-interaction for Dirichlet condition: Theorem 2

Also here homogenized operator $\mathcal{H}^0_\beta$ involves boundary condition (7).

So we introduce $\widetilde{\mathcal{H}}^0_\beta$ with β defined in the theorem.

Given $f \in L_2(\Omega)$, let us denote

$$u^\varepsilon := (\mathcal{H}^\varepsilon - \mathrm{i})^{-1} f, \qquad \widetilde{u}^0 := (\widetilde{\mathcal{H}}^0_\beta - \mathrm{i})^{-1} f, \qquad v^\varepsilon := u^\varepsilon - \widetilde{u}^0.$$

First, we estimate $W_2^1(\Omega^\varepsilon)$-norm of v^ε that solves the problem

$$\left(-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \sum_{j=1}^{2} A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} + A_0 - \mathrm{i} \right) v^\varepsilon = 0 \quad \text{in} \quad \Omega^\varepsilon \setminus \widetilde{\gamma},$$

$$v^\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \qquad v^\varepsilon = -\widetilde{u}^0 \quad \text{on} \quad \partial\theta^\varepsilon, \qquad [v^\varepsilon]_{\widetilde{\gamma}} = 0,$$

$$\left(\frac{\partial}{\partial N^\varepsilon} + a \right) v^\varepsilon = -\left(\frac{\partial}{\partial N^\varepsilon} + a \right) \widetilde{u}^0 \quad \text{on} \quad \partial\theta^\varepsilon_1, \qquad \left[\frac{\partial v^\varepsilon}{\partial \widetilde{N}^0} \right]_{\widetilde{\gamma}} - \beta \widetilde{u}^0 \big|_{\widetilde{\gamma}} = 0.$$

Homogenized delta-interaction for Dirichlet condition: Theorem 2

But v^ε does not satisfy homogeneous Dirichlet condition on $\partial\theta_0^\varepsilon$.

So we add a boundary corrector to v^ε so that the sum vanishes on $\partial\theta_0^\varepsilon$.

Then employing the above boundary value problem,

we shall obtain an integral identity for this sum and estimate its norm.

Previous results: norm resolvent convergence

Norm resolvent convergence was firstly investigated by

- Birman and Suslina,

- V.V. Zhikov and S.E. Pastukhova,

- Griso

- more recently, Kenig, Lin, Shen.

It was shown that the norm resolvent convergence holds true for the elliptic operators with fast oscillating coefficients and

that their resolvents converge to the resolvents of the homogenized operators in the norm resolvent sense.

Moreover, sharp estimates for the rates of convergence in the sense of various operator norms were obtained.

Previous results on Norm resolvent convergence

A natural question appeared: when norm resolvent convergence is valid for other types of the perturbations?

This issue was studied recently for certain perturbations in the boundary homogenization.

Similar results but for the boundary homogenization were established in

- Borisov, D., Cardone, G.: Homogenization of the planar waveguide with frequently alternating boundary conditions. J. Phys. A. **42**, id 365205 (2009)
- D. Borisov, R. Bunoiu, G. Cardone, On a waveguide with frequently alternating boundary conditions: homogenized Neumann condition. Ann. H. Poincaré 11 (2010) 1591-1627.
- D. Borisov, R. Bunoiu, G. Cardone, On a waveguide with an infinite number of small windows. C.R. Math. 349 (2011) 53-56.
- D. Borisov, R. Bunoiu, G. Cardone, Waveguide with non-periodically

Previous results

Such boundary conditions were imposed by partitioning the boundary into small segments where Dirichlet and Robin conditions were imposed in turns.

The homogenized problem involves one of the classical boundary conditions instead of the alternating ones.

For all possible homogenized problems

- the uniform resolvent and

- the estimates for the rates of convergence were proven

for both periodic and non-periodic alternations.

- In periodic cases, asymptotic expansions for the spectra of perturbed operators were constructed.

Previous results

Norm resolvent convergence for problems with a fast periodically oscillating boundary was proven in

O.A. Olejnik, A. S. Shamaev and G. A. Yosifyan, Mathematical problems in elasticity and homogenization. Studies in Mathematics and its Applications, 26, North-Holland, Amsterdam etc. (1992)

S.A. Nazarov, Dirichlet problem in an angular domain with rapidly oscillating boundary: Modeling of the problem and asymptotics of the solution. St. Petersburg Math. J. 19 (2008), 297-326.

Borisov, D., Cardone, G., Faella, L., Perugia, C.: Uniform resolvent convergence for a strip with fast oscillating boundary. J. Diff. Equ. **255**, 4378-4402 (2013)

The most general results in last paper where

various geometries of oscillations and

various boundary conditions on the oscillating boundary were considered.

Previous results

Norm resolvent convergence for periodic perforations (whole of a domain was perforated) in

Marchenko, V.A., Khruslov, E.Ya.: Boundary Value Problems in Domains with Fine-Grained Boundary. Naukova Dumka, Kiev (1974).

Pastukhova, S.E.: Some Estimates from Homogenized Elasticity Problems. Dokl. Math. **73**, 102-106 (2006)

In [1]:

- operator was described by the Helmholtz equation;

- on the boundaries of the holes the Dirichlet condition was imposed.

- holes disappear under the homogenization and made no influence for the homogenized operators.

- no estimates for the rate of convergence were found.

Previous results

In [2]:

- an elliptic operator

- Sizes of the holes and the distances between them are of the same order of smallness.

- On the boundaries of the holes the Neumann condition was imposed.

- estimates for the rate of convergence were established.

Previous results

In

Zhikov, V.V.: Spectral method in homogenization theory. Proc. Steklov Inst. Math. **250**, 85-94 (2005)

- perturbation was defined by rescaling an abstract periodic measure.

- sizes of the holes and the distances between them are of the same smallness order.

- norm resolvent convergence and estimates for the rate of convergence were proven.