Direct and inverse problems for one-dimensional Dirac operators with nonlocal potentials

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Overview

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Part I

Sturm-Liouville operators with nonlocal potentials on the interval
Problem

Consider nonlocal Sturm-Liouville eigenvalue problems of the form

\[(T\psi)(x) \equiv -\frac{d^2\psi(x)}{dx^2} + v(x)\psi(1) = \lambda\psi(x), \quad 0 \leq x \leq 1,\]

with the boundary conditions

\[\psi(0) = \psi'(1) + \langle \psi, v \rangle_{L^2} = 0,\]

where \(v \in L^2(0, 1)\) is the nonlocal potential and \(\lambda \in \mathbb{C}\) is the spectral parameter.

Denote \(\langle \cdot, \cdot \rangle_{L^2}\) by the usual inner product in \(L^2(0, 1)\).
Unperturbed operators

\[ T \psi = -\frac{d^2\psi(x)}{dx^2} + v(x)\psi(1) \]

\[ D(T) = \{ \psi \in W_2^2(0,1) | \psi(0) = \psi'(1) + \langle \psi, v \rangle_{L^2} = 0 \} \]

\[ T_0 \psi = -\frac{d^2\psi(x)}{dx^2} \]

\[ D(T_0) = \{ \psi \in W_2^2(0,1) | \psi(0) = \psi(1) = 0 \} \]

\[ T_1 \psi = -\frac{d^2\psi(x)}{dx^2} \]

\[ D(T_1) = \{ \psi \in W_2^2(0,1) | \psi(0) = \psi'(1) = 0 \} \]
The operators $T_0$ and $T_1$ are self-adjoint and have discrete spectra
$\sigma(T_0) = \{\pi^2 n^2\}_{n \in \mathbb{N}}$ and $\sigma(T_1) = \{\pi^2 (n - \frac{1}{2})^2\}_{n \in \mathbb{N}}$.

**Lemma**

The operator $T$ is self-adjoint and has a discrete spectrum $\{\lambda_n\}_{n \in \mathbb{N}}$, where $\lambda_1 \leq \lambda_2 \leq \ldots$ and each eigenvalue is repeated according to its multiplicity. Moreover, $T$ is a rank-one perturbation of the operator $T_0$ and the spectra of the operators $T$ and $T_0$ weakly interlace, i.e., $\lambda_n \leq \pi^2 n^2 \leq \lambda_{n+1}$ for every $n \in \mathbb{N}$.
Resolvents of $T_0$ and $T_1$

Integral operators

$$(T_j - z^2)^{-1}f(x) = \int_0^1 G_j(x, s; z)f(s)ds, \quad j = 0, 1,$$

Green functions

$G_0(x, s; z) = \frac{1}{z \sin z} \begin{cases} 
\sin zx \sin z(1 - s) & \text{for } s > x, \\
\sin z(1 - x) \sin zs & \text{for } s < x,
\end{cases}$

$G_1(x, s; z) = \frac{1}{z \cos z} \begin{cases} 
\sin zx \cos z(1 - s) & \text{for } s > x, \\
\cos z(1 - x) \sin zs & \text{for } s < x.
\end{cases}$
**Characteristic function**

\[
d(z) = \cos z + \int_0^1 \frac{\sin z s}{z} (v(s) + \overline{v(s)}) ds - \frac{\sin z}{z} \int_0^1 \int_0^1 G_0(x, s; z) v(s) \overline{v(x)} ds dx,
\]

\[
(T - z^2)^{-1} g(x) = (T_0 - z^2)^{-1} g(x) + \frac{z}{\sin zd(z)} \psi(x; z) \langle g, \psi(\cdot, z) \rangle
\]

Let \( \hat{v}_k \) be the k-th Fourier coefficient of the function \( v(x) = \sum_{k=1}^{\infty} \hat{v}_k \sin \pi k x \).

**Other form**

\[
d(z) = \cos z + \frac{\sin z}{2z} \sum_{k=1}^{\infty} \frac{a_k}{z^2 - \pi^2 k^2},
\]

where

\[
a_k = |\hat{v}_k + (-1)^k 2\pi k|^2 - (2\pi k)^2
\]
Theorem

- Every eigenvalue of the operator $T$ is a squared zero of its characteristic function $d$ and, conversely, every squared zero of $d$ is an eigenvalue of $T$. The number $\pi^2 n^2$, $n \in \mathbb{N}$, is an eigenvalue of $T$ if and only if

$$\hat{v}_n = (-1)^{n+1} 2\pi n,$$

and this relation is equivalent to $d(\pi n) = 0$.

- All eigenvalues $z^2$ not in the spectrum of $T_0$ are simple, and simple are the corresponding zeros $z$ of $d$ (except for the case where $z = 0$, which is then a zero of even order of $d$). If $\pi^2 n^2$ for some $n \in \mathbb{N}$ is an eigenvalue of $T$, then this eigenvalue is multiple if and only if

$$\int_0^1 \int_0^1 G_1(x, s; \pi n) \overline{\nu(s)} \nu(x) ds dx = 0,$$

in this and only in this case the number $\pi n$ is a multiple zero of $d$. 
Theorem

- The multiplicity of a non-zero eigenvalue $z^2$ of the operator $T$ equals the order of the corresponding zero $z$ of the characteristic function $d$, and both do not exceed 2. If $z = 0$ is an eigenvalue of $T$, then the order of $z = 0$ as a zero of $d$ is 2.

Asymptotics

The eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ satisfy the asymptotic distribution

$$\sqrt{\lambda_n} = \pi\left(n - \frac{1}{2}\right) + \frac{\mu_n}{n}$$

for some sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\ell_2(\mathbb{N})$. 
Inverse spectral analysis

Given the spectrum \( \sigma(T) \) of an operator find the nonlocal potential \( \nu \).

**Algorithm**

1. Given \( \sigma(T) \), construct the function \( d \) via \( d(z) = \prod_{k \in \mathbb{N}} \frac{\lambda_k - z^2}{\pi^2(k - \frac{1}{2})^2} \).
2. Calculate the values \( d(\pi n), \ n \in \mathbb{N} \).
3. For every \( n \in \mathbb{N} \), solve the quadratic equations \( (\hat{v}_n + (-1)^n2\pi n)^2 = (-1)^n(2\pi n)^2d(\pi n) \) for \( \hat{v}_n \), taking the solution that satisfies the relation \( (-1)^{n+1}\hat{v}_n \leq 2\pi n \).
4. Put \( \nu(x) = \sum_{n \in \mathbb{N}} \hat{v}_n \sin \pi nx \).
Example of a solution to an inverse problem

Let \( \lambda_1 = \pi^2 \) and \( \lambda_n = \pi^2(n - \frac{1}{2})^2 \) for all \( n \geq 2 \). Then

\[
d(z) = \frac{z^2 - \pi^2}{z^2 - \frac{\pi^2}{4}} \cos z,
\]

so that \( d(\pi k) = (-1)^k \frac{k^2 - \frac{1}{4}}{k^2 - \frac{1}{4}} \), \( \hat{v}_1 = 2\pi \) and

\[
\hat{v}_k = (-1)^{k+1} 2\pi k \left( 1 - \sqrt{\frac{k^2 - 1}{k^2 - \frac{1}{4}}} \right) \quad \text{for} \quad k \geq 2.
\]
Part II

First order differential operators with nonlocal potentials on the interval
Boundary value problem

Consider the following nonlocal eigenvalue problems

\[(L\psi)(x) \equiv i \frac{d\psi(x)}{dx} + v(x)\psi_+ = \lambda \psi(x), \quad 0 \leq x \leq l,\]

(1)

with the boundary conditions

\[\psi_- + i \int_0^l \psi(x) \overline{v(x)} dx = 0,\]

(2)

\[\psi_+ := \frac{1}{2} (\psi(l) + \psi(0)), \quad \psi_- := \psi(l) - \psi(0), \quad v \in L^2_2(0, l).\]

The corresponding operator

\[(A\psi)(x) = i \frac{d\psi(x)}{dx} + v(x)\psi_+\]

\[\mathcal{D}(A) = \left\{ \psi \in W^1_2(0, l) : \psi_- + i \int_0^l \psi(x) \overline{v(x)} dx = 0 \right\}\]
First order differential operators with nonlocal potentials on the interval

Direct spectral analysis

The operator $A$ is self-adjoint.

Let $A_-$ and $A_+$ be the differential operators $i \frac{d}{dx}$ on $L_2(0, l)$ with the domains

$$\mathcal{D}(A_-) = \{\psi \in W^1_2(0, l) : \psi_+ = 0\}, \quad \mathcal{D}(A_+) = \{\psi \in W^1_2(0, l) : \psi_- = 0\},$$

respectively. Both these operators are self-adjoint. Their spectra are discrete, the eigenvalues of $A_-$ are $\lambda^{(-)}_n = \frac{2n\pi}{l}$ ($n \in \mathbb{Z}$) and of $A_+$ are $\lambda^{(+)}_n = \frac{\pi(2n-1)}{l}$ ($n \in \mathbb{Z}$) with the corresponding eigenfunctions $\psi^{(-)}_n(x) = e^{-i\lambda^{(-)}_nx}$ and $\psi^{(+)}_n(x) = e^{-i\lambda^{(+)}_nx}$, respectively. The set of eigenfunctions $\{\psi^{(+)}_n : n \in \mathbb{Z}\}$ is a complete orthogonal system in $L_2(0, l)$, and the potential $v \in L_2(0, l)$ can be represented by the Fourier series

$$v(x) = \sum_{n \in \mathbb{Z}} v_n e^{-i(2n-1)\frac{\pi}{l}x},$$

where

$$v_n = \frac{1}{l} \int_0^l v(x) e^{i(2n-1)\frac{\pi}{l}x} dx, \quad n \in \mathbb{Z}.$$
First order differential operators with nonlocal potentials on the interval

A is a rank two perturbation of $A_-$ and a rank one perturbation of $A_+$.  

\[
G_-(x, s; z) = i \frac{e^{-iz(x-s)}}{e^{-izl} - 1} \cdot \begin{cases} 
1 & \text{for } s < x, \\
-1 & \text{for } s > x,
\end{cases}
\]

\[
G_+(x, s; z) = i \frac{e^{-iz(x-s)}}{e^{-izl} + 1} \cdot \begin{cases} 
1 & \text{for } s < x, \\
-1 & \text{for } s > x.
\end{cases}
\]

The resolvent $(A - zI)^{-1}$ is an integral operator and

\[
G(x, s; z) - G_+(x, s; z) = \frac{\varphi(x; z)\overline{\varphi}(s; \bar{z})}{F(z)},
\]

where

\[
F(z) = 2i \frac{1 - e^{-izl}}{1 + e^{-izl}} - 2i \left[ \int_0^l G_+(0, s; z)v(s)ds - \int_0^l G_+(s, 0; z)\overline{v(s)}ds \right] \\
- \int_0^l \int_0^l G_+(x, s; z)v(s)\overline{v(x)}dsdx.
\]
Spectrum

Theorem

1) All eigenvalues of the operator $A$ different from $(2n - 1)\frac{\pi}{l}$, $n \in \mathbb{Z}$, are simple.

2) The number $(2n - 1)\frac{\pi}{l}$, $n \in \mathbb{Z}$, is an eigenvalue of $A$ if and only if

$$v_n \equiv \frac{1}{l} \int_0^l v(x)e^{i\lambda_n^{(+)}} x \, dx = \frac{2i}{l}. $$

3) If $(2n - 1)\frac{\pi}{l}$ is an eigenvalue of $A$, then this eigenvalue has multiplicity 2 if and only if

$$\sum_{k \neq n} \frac{1}{\lambda_k^{(+) - \lambda_n^{(+)}}} \left(v_k - \overline{v_k} - \frac{i}{2} l|v_k|^2\right) = 0.$$

4) The operator $A$ has no eigenvalue with multiplicity exceeding 2.
The characteristic function of the operator $A$ has the following form

$$\chi(\lambda) = -\sin \frac{\lambda l}{2} + \cos \frac{\lambda l}{2} \sum_{n \in \mathbb{Z}} \frac{\alpha_n}{\lambda_n^{(+)} - \lambda}$$

with

$$\alpha_n = -iv_n + i\bar{v}_n - \frac{l}{2} |v_n|^2, \quad n \in \mathbb{Z}.$$ 

The characteristic function $\chi$ of the operator $A$ is an entire function of $\lambda$ and

$$\chi \left( \lambda_n^{(+)} \right) = (-1)^n \left| \frac{l}{2} v_n - i \right|^2, \quad n \in \mathbb{Z}.$$
Theorem

The sequence of eigenvalues of the operator $A$ (counting multiplicities) can be numbered so that

$$
\ldots \leq \lambda_{-n} \leq \ldots \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \leq \ldots
$$

listed in an increasing order satisfies the asymptotic distribution,

$$
\lambda_n = \frac{2\pi}{l} n + \beta_n, \quad \lambda_{-n} = -\frac{2\pi}{l} n + \beta_{-n}, \quad n \in \mathbb{N},
$$

where $\beta_n$ are real values such that

$$
\sum_{k \in \mathbb{Z}} \beta_k^2 < \infty.
$$
Algorithm for solving inverse problem

Let us assume that we know all eigenvalues of the operator $A$, we find the nonlocal potential $v \in L_2(0, l)$.

**Step 1.** Construct the characteristic function $\chi$ as

$$
\chi(\lambda) = -\frac{l}{2}(\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{(\lambda_n - \lambda)(\lambda - \lambda_{-n})}{(\frac{2n\pi}{l})^2}.
$$

**Step 2.** Calculate the values $\chi(\lambda_n^{(+)})$ for all $n \in \mathbb{Z}$, where $\lambda_n^{(+)} = \frac{2n-1}{l}\pi$.

**Step 3.** Solve the quadratic equation for $v_n$

$$
\chi\left(\lambda_n^{(+)}\right) = (-1)^n \left| \frac{l}{2} v_n - i \right|^2.
$$

**Step 4.** Write the potential $v(x) = \sum_{n \in \mathbb{Z}} v_ne^{-i\lambda_n^{(+)}x}$.
Example for solving inverse problem

Let $\lambda_1 = \frac{1}{2}$ and $\lambda_n = n$ for $n \neq 1$ be the eigenvalues of the operator $A$ and let $l = 2\pi$. The characteristic function $\chi$, in this case, is the following

$$\chi(\lambda) = -\sin (\pi \lambda) \frac{\lambda - \frac{1}{2}}{\lambda - 1}.$$

For $\lambda_n^{(+)} = n - \frac{1}{2}$ calculate the values $\chi(\lambda_n^{(+)}) = (-1)^n \frac{n-1}{n-\frac{3}{2}}$. We solve the quadratic equation

$$(-1)^n |\pi v_n - i|^2 = (-1)^n \frac{n-1}{n-\frac{3}{2}},$$

which is equivalent to

$$|\pi v_n - i|^2 = \frac{n-1}{n-\frac{3}{2}},$$

from which we compute the Fourier coefficients of the potential $v$

$$v_n = -\frac{i}{2\pi} \left( \left| n - \frac{3}{2} \right| + \sqrt{(n-1) \left( n - \frac{3}{2} \right)} \right)^{-1}.$$
Part III

Dirac systems with nonlocal potentials on the interval
The following spectral problem for the Dirac system with the nonlocal potentials

\[
\begin{align*}
&i \frac{d\psi_1(x)}{dx} + v_1(x)\psi^+ = \lambda \psi_1(x), \\
&-i \frac{d\psi_2(x)}{dx} + v_2(x)\psi^+ = \lambda \psi_2(x),
\end{align*}
\]

where

\[
\psi_1, \psi_2 \in W^1_2(0, b), \quad v_1, v_2 \in L^2(0, b), \quad \psi^+ := \frac{1}{2} (\psi_1(b) + \psi_2(b))
\]

with the boundary conditions

\[
\psi_1(0) = \psi_2(0),
\]

\[
\psi_1(b) - \psi_2(b) + i (\langle \psi_1, v_1 \rangle + \langle \psi_2, v_2 \rangle) = 0
\]

is equivalent to the problem (1)–(2).
Theorem

Moreover, the corresponding operator $A$ defined by

$$(A\psi)(x) = B \frac{d\psi(x)}{dx} + V(x)\psi^+$$

where

$$B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad V(x) = \begin{pmatrix} 0 & v_1(x) \\ v_2(x) & 0 \end{pmatrix}, \quad \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad \psi^+ = \begin{pmatrix} \psi^+ \\ \psi^+ \end{pmatrix},$$

with the domain

$$\mathcal{D}(A) = \left\{ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \psi_1, \psi_2 \in W^1_2(0, b) : \psi_1(0) = \psi_2(0), \psi_1(b) - \psi_2(b) + i (\langle \psi_1, v_1 \rangle + \langle \psi_2, v_2 \rangle) = 0, \right\}$$

is self-adjoint.
Proof

We consider the problem on the interval $[0, 2b]$ in the following way

$$
\psi_2(b - x) \quad \psi_1(x - b)
$$

0 \quad b \quad 2b

We define the functions

$$
\psi(x) = \begin{cases} 
\psi_1(x - b), & b \leq x \leq 2b, \\
\psi_2(b - x), & 0 \leq x \leq b,
\end{cases}
$$

and

$$
v(x) = \begin{cases} 
v_1(x - b), & b \leq x \leq 2b, \\
v_2(b - x), & 0 \leq x \leq b.
\end{cases}
$$

Then

$$
\psi^+ = \frac{1}{2} (\psi(2b) + \psi(0)),
$$

which is equal to $\psi_+ = \frac{1}{2} (\psi(l) + \psi(0))$ for $l = 2b$. In fact, we write the Dirac system with nonlocal potentials as the eigenvalue problem for the first order differential operator, substituting $l = 2b$. 

The nonlocal potentials $v_1$ and $v_2$ can be represented by the Fourier series

$$v_j(x) = \sum_{n \in \mathbb{Z}} v_n^{(j)} \psi_j(x), \quad 0 \leq x \leq b, \quad j = 1, 2,$$

and, respectively,

$$v_n^{(j)} = \frac{1}{b} \int_0^b v_j(x) \overline{\psi_j(x)} dx, \quad j = 1, 2.$$
Description of the spectrum

**Theorem**

1) All eigenvalues of the operator $A$ different from $(n - \frac{1}{2}) \frac{\pi}{b}$, $n \in \mathbb{Z}$, are simple.

2) The number $(n - \frac{1}{2}) \frac{\pi}{b}$, $n \in \mathbb{Z}$, is an eigenvalue of $A$ if and only if

$$\tilde{v}_n = \frac{2}{b} (-1)^{(n+1)} \quad (\tilde{v}_n := v_n^{(1)} + v_n^{(2)}).$$

3) If $(n - \frac{1}{2}) \frac{\pi}{b}$ is an eigenvalue of $A$, then this eigenvalue has multiplicity 2 if and only if

$$\sum_{k \neq n} \frac{1}{\lambda_k^{(+)} - \lambda_n^{(+)}} \left( (-1)^{k+1} \tilde{v}_k + (-1)^{k+1} \tilde{v}_k - \frac{b}{2} |\tilde{v}_k|^2 \right) = 0.$$  

4) The operator $A$ has no eigenvalue with multiplicity exceeding 2.
The characteristic function of the operator $\mathcal{A}$ has the form

$$\chi(\lambda) = -\sin(\lambda b) + \cos(\lambda b) \sum_{n \in \mathbb{Z}} \frac{\alpha_n}{\lambda_n^{(+)} - \lambda},$$

where

$$\alpha_n = \frac{1}{2} (-1)^{n+1} \left( v_n^{(1)} + v_n^{(2)} \right) + \frac{1}{2} (-1)^{n+1} \left( \overline{v}_n^{(1)} + \overline{v}_n^{(2)} \right) - \frac{b}{4} \left| v_n^{(1)} + v_n^{(2)} \right|^2,$$

and $\lambda_n^{(+)} = (n - \frac{1}{2}) \frac{\pi}{b}$. Moreover, we infer the following equation

$$\chi \left( \lambda_n^{(+)} \right) = (-1)^n \left| \frac{b}{2} (-1)^{n+1} \left( v_n^{(1)} + v_n^{(2)} \right) - 1 \right|^2.$$
Inverse problem

Step 1. Knowing all eigenvalues $\lambda_n$ of the operator $A$, we construct the characteristic function $\chi$:

$$
\chi(\lambda) = -b(\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{(\lambda_n - \lambda)(\lambda - \lambda_{-n})}{(n\pi b)^2}.
$$

Step 2. We calculate the values $\chi(\lambda_n^{(+)})$ for all $n \in \mathbb{Z}$, where $\lambda_n^{(+)} = (n - \frac{1}{2}) \frac{\pi}{b}$.

Step 3. Solve the quadratic equation for $v_n$

$$
\chi \left( \lambda_n^{(+)} \right) = (-1)^n |bv_n - i|^2.
$$

Step 4. Using potential $v$, we find the potentials $v_1, v_2$ by reducing procedure.
Example for solving inverse problem

Let $\lambda_1 = \frac{1}{2}$ and $\lambda_n = n$ for $n \neq 1$ be the eigenvalues of the operator $A$ and let $b = \pi$. The characteristic function $\chi$, in this case, is the following

$$\chi(\lambda) = -\sin(\pi \lambda) \frac{\lambda - \frac{1}{2}}{\lambda - 1}.$$ 

For $\lambda_n^{(+)} = n - \frac{1}{2}$ calculate the values $\chi(\lambda_n^{(+)}) = (-1)^n \frac{n-1}{n-\frac{3}{2}}$. We solve the quadratic equation

$$|\pi v_n - i|^2 = \frac{n-1}{n-\frac{3}{2}},$$

and get

$$v_n = -\frac{i}{2\pi} \left( \left| n - \frac{3}{2} \right| + \sqrt{(n - 1) \left( n - \frac{3}{2} \right)} \right)^{-1}.$$ 

Then

$$v_n^{(j)} = \frac{(-1)^n}{2\pi} \left( \left| n - \frac{3}{2} \right| + \sqrt{(n - 1) \left( n - \frac{3}{2} \right)} \right)^{-1} \quad j = 1, 2.$$
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Thank you for your attention