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Direct and inverse problems for one-dimensional Dirac operators with nonlocal potentials

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Overview

Part I

Sturm-Liouville operators with nonlocal potentials on the interval

Sturm-Liouville eigenvalue problems

Problem

Consider nonlocal Sturm-Liouville eigenvalue problems of the form

$$(T\psi)(x) \equiv -\frac{d^2\psi(x)}{dx^2} + v(x)\psi(1) = \lambda\psi(x), \quad 0 \le x \le 1,$$

with the boundary conditions

$$\psi(0) = \psi'(1) + \langle \psi, v \rangle_{L_2} = 0,$$

where $v \in L_2(0,1)$ is the nonlocal potential and $\lambda \in \mathbb{C}$ is the spectral parameter.

Denote $\langle \cdot, \cdot \rangle_{L_2}$ by the usual inner product in $L_2(0,1)$.

Unperturbed operators

$$T\psi = -\frac{d^2\psi(x)}{dx^2} + v(x)\psi(1)$$

$$\mathcal{D}(T) = \{\psi \in W_2^2(0,1) | \psi(0) = \psi'(1) + \langle \psi, v \rangle_{L_2} = 0\}$$

$$T_0\psi = -\frac{d^2\psi(x)}{dx^2}$$

$$\mathcal{D}(T_0) = \{\psi \in W_2^2(0,1) | \psi(0) = \psi(1) = 0\}$$

$$T_1\psi = -\frac{d^2\psi(x)}{dx^2}$$

$$\mathcal{D}(T_1) = \{\psi \in W_2^2(0,1) | \psi(0) = \psi'(1) = 0\}$$

Spectrum

The operators T_0 and T_1 are self-adjoint and have discrete spectra $\sigma(T_0) = \{\pi^2 n^2\}_{n\in\mathbb{N}}$ and $\sigma(T_1) = \{\pi^2(n-\frac{1}{2})^2\}_{n\in\mathbb{N}}$.

Lemma

The operator T is self-adjoint and has a discrete spectrum $\{\lambda_n\}_{n\in\mathbb{N}}$, where $\lambda_1 \le \lambda_2 \le \dots$ and each eigenvalue is repeated according to its multiplicity. Moreover, T is a rank-one perturbation of the operator T_0 and the spectra of the operators T and T_0 weakly interlace, i.e., $\lambda_n \le \pi^2 n^2 \le \lambda_{n+1}$ for every $n \in \mathbb{N}$.

Resolvents of T_0 and T_1

Integral operators

$$(T_j - z^2)^{-1} f(x) = \int_0^1 G_j(x, s; z) f(s)\,ds, \quad j = 0, 1,$$

Green functions

$$G_0(x, s; z) = \frac{1}{z \sin z} \begin{cases} \sin zx \sin z(1-s) & \text{for} \quad s > x, \\ \sin z(1-x) \sin zs & \text{for} \quad s < x, \end{cases}$$

$$G_1(x, s; z) = \frac{1}{z \cos z} \begin{cases} \sin zx \cos z(1-s) & \text{for} \quad s > x, \\ \cos z(1-x) \sin zs & \text{for} \quad s < x. \end{cases}$$

Characteristic function

Characteristic function

$$d(z) = \cos z + \int_0^1 \frac{\sin zs}{z}(v(s) + \overline{v(s)})ds - \frac{\sin z}{z}\int_0^1\int_0^1 G_0(x, s; z)v(s)\overline{v(x)}dsdx,$$

$$(T - z^2)^{-1}g(x) = (T_0 - z^2)^{-1}g(x) + \frac{z}{\sin z d(z)}\psi(x; z)\langle g, \psi(\cdot, \overline{z})\rangle$$

Let $\hat{v}_k$ be the k-th Fourier coefficient of the function $v(x) = \sum_{k=1}^{\infty} \hat{v}_k \sin \pi kx$.

Other form

$$d(z) = \cos z + \frac{\sin z}{2z}\sum_{k=1}^{\infty}\frac{a_k}{z^2 - \pi^2 k^2},$$

where

$$a_k = |\hat{v}_k + (-1)^k 2\pi k|^2 - (2\pi k)^2$$

Direct spectral analysis

Theorem

- Every eigenvalue of the operator T is a squared zero of its characteristic function d and, conversely, every squared zero of d is an eigenvalue of T. The number $\pi^2 n^2$, $n \in \mathbb{N}$, is an eigenvalue of T if and only if

$$\hat{v}_n = (-1)^{n+1} 2\pi n,$$

  and this relation is equivalent to $d(\pi n) = 0$.
- All eigenvalues z^2 not in the spectrum of T_0 are simple, and simple are the corresponding zeros z of d (except for the case where $z = 0$, which is then a zero of even order of d). If $\pi^2 n^2$ for some $n \in \mathbb{N}$ is an eigenvalue of T, then this eigenvalue is multiple if and only if

$$\int_0^1 \int_0^1 G_1(x, s; \pi n) v(s) \overline{v(x)} ds dx = 0,$$

  in this and only in this case the number πn is a multiple zero of d.

Direct spectral analysis

Theorem

- The multiplicity of a non-zero eigenvalue z^2 of the operator T equals the order of the corresponding zero z of the characteristic function d, and both do not exceed 2. If $z = 0$ is an eigenvalue of T, then the order of $z = 0$ as a zero of d is 2.

Asymptotics

The eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ satisfy the asymptotic distribution

$$\sqrt{\lambda_n} = \pi(n - \frac{1}{2}) + \frac{\mu_n}{n}$$

for some sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\ell_2(\mathbb{N})$.

Inverse spectral analysis

Given the spectrum $\sigma(T)$ of an operator find the nonlocal potential v.

Algorithm

1. Given $\sigma(T)$, construct the function d via $d(z) = \prod_{k\in\mathbb{N}} \frac{\lambda_k - z^2}{\pi^2(k-\frac{1}{2})^2}$.
2. Calculate the values $d(\pi n)$, $n \in \mathbb{N}$.
3. For every $n \in \mathbb{N}$, solve the quadratic equations $(\hat{v}_n + (-1)^n 2\pi n)^2 = (-1)^n (2\pi n)^2 d(\pi n)$ for $\hat{v}_n$, taking the solution that satisfies the relation $(-1)^{n+1}\hat{v}_n \le 2\pi n$.
4. Put $v(x) = \sum_{n\in\mathbb{N}} \hat{v}_n \sin \pi n x$.

Example of a solution to an inverse problem

Example

Let $\lambda_1 = \pi^2$ and $\lambda_n = \pi^2(n - \frac{1}{2})^2$ for all $n \geq 2$. Then

$$d(z) = \frac{z^2 - \pi^2}{z^2 - \frac{\pi^2}{4}} \cos z,$$

so that $d(\pi k) = (-1)^k \frac{k^2-1}{k^2-\frac{1}{4}}, \hat{v}_1 = 2\pi$ and

$$\hat{v}_k = (-1)^{k+1} 2\pi k \left(1 - \sqrt{\frac{k^2-1}{k^2-\frac{1}{4}}}\right) \quad \text{for} \quad k \geq 2.$$

Part II

First order differential operators with nonlocal potentials on the interval

Boundary value problem

Consider the following nonlocal eigenvalue problems

$$(L\psi)(x) \equiv i\frac{d\psi(x)}{dx} + v(x)\psi_+ = \lambda\psi(x), \quad 0 \leq x \leq l, \tag{1}$$

with the boundary conditions

$$\psi_- + i\int_0^l \psi(x)\overline{v(x)}dx = 0, \tag{2}$$

$$\psi_+ := \frac{1}{2}\left(\psi(l) + \psi(0)\right), \quad \psi_- := \psi(l) - \psi(0), \quad v \in L_2(0,l).$$

The corresponding operator

$$(A\psi)(x) = i\frac{d\psi(x)}{dx} + v(x)\psi_+$$

$$\mathcal{D}(A) = \left\{\psi \in W_2^1(0,l) : \psi_- + i\int_0^l \psi(x)\overline{v(x)}dx = 0\right\}$$

Direct spectral analysis

The operator A is self-adjoint.

Let A_- and A_+ be the differential operators $i\frac{d}{dx}$ on $L_2(0,l)$ with the domains

$$\mathcal{D}(A_-) = \{\psi \in W_2^1(0,l) : \psi_- = 0\}, \quad \mathcal{D}(A_+) = \{\psi \in W_2^1(0,l) : \psi_+ = 0\},$$

respectively. Both these operators are self-adjoint. Their spectra are discrete, the eigenvalues of A_- are $\lambda_n^{(-)} = \frac{2n\pi}{l}$ $(n \in \mathbb{Z})$ and of A_+ are $\lambda_n^{(+)} = \frac{\pi(2n-1)}{l}$ $(n \in \mathbb{Z})$ with the corresponding eigenfunctions $\psi_n^{(-)}(x) = e^{-i\lambda_n^{(-)}x}$ and $\psi_n^{(+)}(x) = e^{-i\lambda_n^{(+)}x}$, respectively. The set of eigenfunctions $\left\{\psi_n^{(+)} : n \in \mathbb{Z}\right\}$ is a complete orthogonal system in $L_2(0,l)$, and the potential $v \in L_2(0,l)$ can be represented by the Fourier series

$$v(x) = \sum_{n\in\mathbb{Z}} v_n e^{-i(2n-1)\frac{\pi}{l}x},$$

where

$$v_n = \frac{1}{l}\int_0^l v(x) e^{i(2n-1)\frac{\pi}{l}x} dx, \quad n \in \mathbb{Z}.$$

A is a rank two perturbation of A_- and a rank one perturbation of A_+.

$$G_-(x,s;z) = i\frac{e^{-iz(x-s)}}{e^{-izl}-1}\cdot\begin{cases} 1 & \text{for} \quad s<x, \\ e^{-izl} & \text{for} \quad s>x, \end{cases}$$

$$G_+(x,s;z) = i\frac{e^{-iz(x-s)}}{e^{-izl}+1}\cdot\begin{cases} -1 & \text{for} \quad s<x, \\ e^{-izl} & \text{for} \quad s>x. \end{cases}$$

The resolvent $(A-zI)^{-1}$ is an integral operator and

$$G(x,s;z) - G_+(x,s;z) = \frac{\varphi(x;z)\bar{\varphi}(s;\bar{z})}{F(z)},$$

where

$$F(z) = 2i\frac{1-e^{-izl}}{1+e^{-izl}} - 2i\left[\int_0^l G_+(0,s;z)v(s)ds - \int_0^l G_+(s,0;z)\overline{v(s)}ds\right]$$
$$-\int_0^l\int_0^l G_+(x,s;z)v(s)\overline{v(x)}dsdx.$$

Spectrum

Theorem

1) All eigenvalues of the operator A different from $(2n-1)\frac{\pi}{l}$, $n \in \mathbb{Z}$, are simple.
2) The number $(2n-1)\frac{\pi}{l}$, $n \in \mathbb{Z}$, is an eigenvalue of A if and only if

$$v_n \equiv \frac{1}{l}\int_0^l v(x)e^{i\lambda_n^{(+)}x}dx = \frac{2i}{l}.$$

3) If $(2n-1)\frac{\pi}{l}$ is an eigenvalue of A, then this eigenvalue has multiplicity 2 if and only if

$$\sum_{k \neq n} \frac{1}{\lambda_k^{(+)} - \lambda_n^{(+)}}\left(v_k - \overline{v_k} - \frac{i}{2}l|v_k|^2\right) = 0.$$

4) The operator A has no eigenvalue with multiplicity exceeding 2.

Characteristic function

The characteristic function of the operator A has the following form

$$\chi(\lambda) = -\sin\frac{\lambda l}{2} + \cos\frac{\lambda l}{2}\sum_{n\in\mathbb{Z}}\frac{\alpha_n}{\lambda_n^{(+)} - \lambda}$$

with

$$\alpha_n = -iv_n + i\overline{v_n} - \frac{l}{2}|v_n|^2, \quad n \in \mathbb{Z}.$$

The characteristic function χ of the operator A is an entire function of λ and

$$\chi\left(\lambda_n^{(+)}\right) = (-1)^n\left|\frac{l}{2}v_n - i\right|^2, \quad n \in \mathbb{Z}.$$

Asymptotics

Theorem

The sequence of eigenvalues of the operator A (counting multiplicities) can number so that

$$\ldots \leq \lambda_{-n} \leq \ldots \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \leq \ldots$$

listed in an increasing order satisfies the asymptotic distribution,

$$\lambda_n = \frac{2\pi}{l} n + \beta_n, \quad \lambda_{-n} = -\frac{2\pi}{l} n + \beta_{-n}, \quad n \in \mathbb{N},$$

where β_n are real values such that

$$\sum_{k \in \mathbb{Z}} \beta_k^2 < \infty.$$

Algorithm for solving inverse problem

Let us assume that we know all eigenvalues of the operator A, we find the nonlocal potential $v \in L_2(0, l)$.

Step 1. Construct the characteristic function χ as

$$\chi(\lambda) = -\frac{l}{2}(\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{(\lambda_n - \lambda)(\lambda - \lambda_{-n})}{\left(\frac{2n\pi}{l}\right)^2}.$$

Step 2. Calculate the values $\chi(\lambda_n^{(+)})$ for all $n \in \mathbb{Z}$, where $\lambda_n^{(+)} = \frac{2n-1}{l}\pi$.

Step 3. Solve the quadratic equation for v_n

$$\chi\left(\lambda_n^{(+)}\right) = (-1)^n \left|\frac{l}{2} v_n - i\right|^2.$$

Step 4. Write the potential $v(x) = \sum_{n\in\mathbb{Z}} v_n e^{-i\lambda_n^{(+)} x}$.

Example for solving inverse problem

Let $\lambda_1 = \frac{1}{2}$ and $\lambda_n = n$ for $n \neq 1$ be the eigenvalues of the operator A and let $l = 2\pi$. The characteristic function χ, in this case, is the following

$$\chi(\lambda) = -\sin(\pi\lambda)\frac{\lambda - \frac{1}{2}}{\lambda - 1}.$$

For $\lambda_n^{(+)} = n - \frac{1}{2}$ calculate the values $\chi(\lambda_n^{(+)}) = (-1)^n \frac{n-1}{n-\frac{3}{2}}$. We solve the quadratic equation

$$(-1)^n \left|\pi v_n - i\right|^2 = (-1)^n \frac{n-1}{n-\frac{3}{2}},$$

which is equivalent to

$$\left|\pi v_n - i\right|^2 = \frac{n-1}{n-\frac{3}{2}},$$

from which we compute the Fourier coefficients of the potential v

$$v_n = -\frac{i}{2\pi}\left(\left|n - \frac{3}{2}\right| + \sqrt{\left(n-1\right)\left(n-\frac{3}{2}\right)}\right)^{-1}.$$

Part III

Dirac systems with nonlocal potentials on the interval

Spectral problems

Theorem

The following spectral problem for the Dirac system with the nonlocal potentials

$$\begin{cases} i\frac{d\psi_1(x)}{dx} + v_1(x)\psi^+ = \lambda\psi_1(x), \\ -i\frac{d\psi_2(x)}{dx} + v_2(x)\psi^+ = \lambda\psi_2(x), \end{cases} \quad 0 \le x \le b, \quad (0 < b < \infty),$$

where

$$\psi_1, \psi_2 \in W_2^1(0, b), \quad v_1, v_2 \in L_2(0, b), \quad \psi^+ := \frac{1}{2}(\psi_1(b) + \psi_2(b))$$

with the boundary conditions

$$\psi_1(0) = \psi_2(0),$$

$$\psi_1(b) - \psi_2(b) + i(\langle\psi_1, v_1\rangle + \langle\psi_2, v_2\rangle) = 0$$

is equivalent to the problem (1)–(2).

Theorem

Moreover, the corresponding operator $\mathcal{A}$ defined by

$$(\mathcal{A}\Psi)(x) = B\frac{d\Psi(x)}{dx} + V(x)\Psi^+$$

where

$$B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad V(x) = \begin{pmatrix} 0 & v_1(x) \\ v_2(x) & 0 \end{pmatrix}, \quad \Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad \Psi^+ = \begin{pmatrix} \psi^+ \\ \psi^+, \end{pmatrix},$$

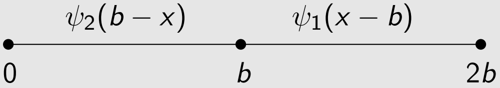
with the domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \psi_1, \psi_2 \in W_2^1(0,b) : \psi_1(0) = \psi_2(0), \right.$$
$$\left. \psi_1(b) - \psi_2(b) + i\left(\langle \psi_1, v_1\rangle + \langle \psi_2, v_2\rangle\right) = 0, \right\}$$

is self-adjoint.

Proof

We consider the problem on the interval $[0, 2b]$ in the following way



We define the functions

$$\psi(x) = \begin{cases} \psi_1(x-b), & b \leq x \leq 2b, \\ \psi_2(b-x), & 0 \leq x \leq b, \end{cases}$$

and

$$v(x) = \begin{cases} v_1(x-b), & b \leq x \leq 2b, \\ v_2(b-x), & 0 \leq x \leq b. \end{cases}$$

Then

$$\psi^+ = \frac{1}{2}\left(\psi(2b) + \psi(0)\right),$$

which is equal to $\psi_+ = \frac{1}{2}\left(\psi(l) + \psi(0)\right)$ for $l = 2b$. In fact, we write the Dirac system with nonlocal potentials as the eigenvalue problem for the first order differential operator, substituting $l = 2b$.

Fourier series

$$\lambda_n^{(+)} = \left(n - \frac{1}{2}\right)\frac{\pi}{b}, \quad \psi_1(x) = e^{-i\lambda_n^{(+)}x}. \quad \psi_2(x) = e^{i\lambda_n^{(+)}x}$$

The nonlocal potentials v_1 and v_2 can be represented by the Fourier series

$$v_j(x) = \sum_{n\in\mathbb{Z}} v_n^{(j)}\psi_j(x), \quad 0 \le x \le b, \quad j = 1, 2,$$

and, respectively,

$$v_n^{(j)} = \frac{1}{b}\int_0^b v_j(x)\overline{\psi_j(x)}dx, \quad j = 1, 2.$$

Description of the spectrum

Theorem

1) All eigenvalues of the operator $\mathcal{A}$ different from $(n-\frac{1}{2})\frac{\pi}{b}$, $n \in \mathbb{Z}$, are simple.
2) The number $(n-\frac{1}{2})\frac{\pi}{b}$, $n \in \mathbb{Z}$, is an eigenvalue of $\mathcal{A}$ if and only if
$$\tilde{v}_n = \frac{2}{b}(-1)^{(n+1)} \quad (\tilde{v}_n := v_n^{(1)} + v_n^{(2)}).$$
3) If $(n-\frac{1}{2})\frac{\pi}{b}$ is an eigenvalue of $\mathcal{A}$, then this eigenvalue has multiplicity 2 if and only if
$$\sum_{k\neq n} \frac{1}{\lambda_k^{(+)} - \lambda_n^{(+)}} \left((-1)^{k+1}\tilde{v}_k + (-1)^{k+1}\overline{\tilde{v}_k} - \frac{b}{2}\left|\tilde{v}_k\right|^2 \right) = 0.$$
4) The operator $\mathcal{A}$ has no eigenvalue with multiplicity exceeding 2.

Characteristic function

The characteristic function of the operator $\mathcal{A}$ has the form

$$\chi(\lambda) = -\sin(\lambda b) + \cos(\lambda b) \sum_{n\in\mathbb{Z}} \frac{\alpha_n}{\lambda_n^{(+)} - \lambda},$$

where

$$\alpha_n = \frac{1}{2}(-1)^{n+1}\left(v_n^{(1)} + v_n^{(2)}\right) + \frac{1}{2}(-1)^{n+1}\left(\overline{v}_n^{(1)} + \overline{v}_n^{(2)}\right) - \frac{b}{4}\left|v_n^{(1)} + v_n^{(2)}\right|^2,$$

and $\lambda_n^{(+)} = (n - \frac{1}{2})\frac{\pi}{b}$. Moreover, we infer the following equation

$$\chi\left(\lambda_n^{(+)}\right) = (-1)^n \left|\frac{b}{2}(-1)^{n+1}\left(v_n^{(1)} + v_n^{(2)}\right) - 1\right|^2.$$

Inverse problem

Step 1. Knowing all eigenvalues λ_n of the operator $\mathcal{A}$, we construct the characteristic function χ:

$$\chi(\lambda) = -b(\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{(\lambda_n - \lambda)(\lambda - \lambda_{-n})}{\left(\frac{n\pi}{b}\right)^2}.$$

Step 2. We calculate the values $\chi(\lambda_n^{(+)})$ for all $n \in \mathbb{Z}$, where $\lambda_n^{(+)} = (n - \frac{1}{2})\frac{\pi}{b}$.

Step 3. Solve the quadratic equation for v_n

$$\chi\left(\lambda_n^{(+)}\right) = (-1)^n \left|b v_n - i\right|^2.$$

Step 4. Using potential v, we find the potentials v_1, v_2 by reducing procedure.

Example for solving inverse problem

Let $\lambda_1 = \frac{1}{2}$ and $\lambda_n = n$ for $n \neq 1$ be the eigenvalues of the operator $\mathcal{A}$ and let $b = \pi$. The characteristic function χ, in this case, is the following

$$\chi(\lambda) = -\sin(\pi\lambda)\frac{\lambda - \frac{1}{2}}{\lambda - 1}.$$

For $\lambda_n^{(+)} = n - \frac{1}{2}$ calculate the values $\chi(\lambda_n^{(+)}) = (-1)^n \frac{n-1}{n-\frac{3}{2}}$. We solve the quadratic equation

$$|\pi v_n - i|^2 = \frac{n-1}{n-\frac{3}{2}},$$

and get

$$v_n = -\frac{i}{2\pi}\left(\left|n - \frac{3}{2}\right| + \sqrt{\left(n-1\right)\left(n - \frac{3}{2}\right)}\right)^{-1}.$$

Then

$$v_n^{(j)} = \frac{(-1)^n}{2\pi}\left(\left|n - \frac{3}{2}\right| + \sqrt{\left(n-1\right)\left(n - \frac{3}{2}\right)}\right)^{-1} \quad j = 1, 2.$$

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Thank you for your attention