Spectra of periodic quantum graphs: more than one would expect

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Motivation

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- in two- or more dimensional systems the number of open gaps is \textit{always finite} by the Bethe-Sommerfled conjecture
- \textit{band edges} are reached at the \textit{boundary} of the Brillouin zone or in its \textit{center}
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![Diagram](image.png)

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To show that the spectrum may be even *pure point* we consider our first example which concerns a *chain graph* in a *magnetic field*, in general nonconstant.
The magnetic chain

To be specific, the chain graph will look as follows

\[ \psi_j \mapsto -D^2 \psi_j \] on each graph link, where

\[ D = -i \nabla - A, \] and for definiteness we assume \( \delta \)-coupling in the vertices, i.e. the domain consists of functions from \( H^2_{\text{loc}}(\Gamma) \) satisfying

\[ \psi_i(0) = \psi_j(0) =: \psi(0), \quad i, j \in \mathbb{N}, \sum_i D \psi_i(0) = \alpha \psi(0), \] where

\( \mathbb{N} = \{1, 2, \ldots, n\} \) is the index set numbering the edges – in our case \( n = 4 \) – and \( \alpha \in \mathbb{R} \) is the coupling constant.
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The Hamiltonian is *magnetic Laplacian*, $\psi_j \mapsto -\mathcal{D}^2 \psi_j$ on each graph link, where $\mathcal{D} := -i \nabla - A$.
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\[
\begin{align*}
\cdots & \quad 0 \quad A_j-1 \quad \pi \quad 0 \quad A_j \quad \pi \quad 0 \quad A_j+1 \quad \pi \quad 0 \quad \cdots \\
& \quad e_{j-1}^U \quad e_j^U \quad e_{j+1}^U \\
& \quad e_{j-1}^L \quad e_j^L \quad e_{j+1}^L \\
& \quad v_{j-1} \quad v_j \quad v_{j+1} \quad v_{j+2} \\
& \quad \vdots
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The Hamiltonian is \textit{magnetic Laplacian}, \( \psi_j \mapsto -D^2 \psi_j \) on each graph link, where \( D := -i \nabla - A \), and for definiteness we assume \textit{δ-coupling} in the vertices, i.e. the domain consists of functions from \( H^2_{\text{loc}}(\Gamma) \) satisfying

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\psi_i(0) = \psi_j(0) =: \psi(0), \quad i, j \in n, \quad \sum_{i=1}^{n} D \psi_i(0) = \alpha \psi(0),
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![Diagram of a magnetic chain graph]

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This is a particular case of the general conditions that make the operator self-adjoint [Kostrykin-Schrader’03]
We write \( \psi_L(x) = e^{-iAx}(C_L^+ e^{ikx} + C_L^- e^{-ikx}) \) for \( x \in [-\pi/2, 0] \) and energy \( E := k^2 \neq 0 \), and similarly for the other three components.
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\[ \sin k\pi \cos A\pi (e^{2i\theta} - 2\xi(k)e^{i\theta} + 1) = 0 \]

with $\xi(k) := \cos A\pi (\cos k\pi + \alpha^4k \sin k\pi)$, for any $k \in \mathbb{R} \cup i\mathbb{R}\{0\}$ and the discriminant equal to $D = 4(\xi(k)^2 - 1)$.

Apart from the cases $A - 1/2 \in \mathbb{Z}$ and $k \in \mathbb{N}$ we have $k^2 \in \sigma(-\Delta')$ iff the condition $|\xi(k)| \leq 1$ is satisfied.
Floquet-Bloch analysis of the fully periodic case

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The functions have to be matched through (a) the $\delta$-coupling and (b) Floquet-Bloch conditions. This equation for the phase factor $e^{i\theta}$,

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Apart from the cases $A - \frac{1}{2} \in \mathbb{Z}$ and $k \in \mathbb{N}$ we have $k^2 \in \sigma(-\Delta_\alpha)$ iff the condition $|\xi(k)| \leq 1$ is satisfied.
The picture refers to $A = 0$ with $\eta(z) := 4\xi(\sqrt{z})$ and $\gamma = \alpha$. 
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For $A - \frac{1}{2} \notin \mathbb{Z}$ the situation is similar, just the width of the band changes to $4 \cos A \pi$, on the other hand, for $A - \frac{1}{2} \in \mathbb{Z}$ it shrinks to a line

In picture: determining the spectral bands
The idea was put forward by physicists – *Alexander* and *de Gennes* – and later treated rigorously in [Cattaneo’97], [E’97], and [Pankrashkin’13]
Duality

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We exclude possible Dirichlet eigenvalues from our considerations assuming $k \in \mathcal{K} := \{z : \text{Im } z \geq 0 \land z \notin \mathbb{Z}\}$. On the one hand, we have the differential equation

$$(-\Delta_{\alpha,A} - k^2) \begin{pmatrix} \psi(x, k) \\ \varphi(x, k) \end{pmatrix} = 0$$

with the components referring to the upper and lower part of $\Gamma$, ...
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$$\psi_{j+1}(k) + \psi_{j-1}(k) = \xi_j(k) \psi_j(k), \quad k \in \mathcal{K},$$

where $\psi_j(k) := \psi(j\pi,k)$ and $\xi(k)$ was introduced above, $\xi_j$ corresponding the coupling $\alpha_j$. The two equations are intimately related.
Duality, continued

**Theorem**

Let $\alpha_j \in \mathbb{R}$, then any solution \( \begin{pmatrix} \psi(\cdot, k) \\ \varphi(\cdot, k) \end{pmatrix} \) with $k^2 \in \mathbb{R}$ and $k \in \mathbb{R}$ satisfies the difference equation, and conversely, the latter defines via

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\begin{pmatrix} \psi(x, k) \\ \varphi(x, k) \end{pmatrix} = e^{\mp iA(x-j\pi)} \left[ \psi_j(k) \cos k(x - j\pi) \\
+ (\psi_{j+1}(k)e^{\pm iA\pi} - \psi_j(k) \cos k\pi) \frac{\sin k(x - j\pi)}{\sin k\pi} \right], \quad x \in (j\pi, (j + 1)\pi),
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solutions to the former satisfying the $\delta$-coupling conditions. In addition, the former belongs to $L^p(\Gamma)$ if and only if \( \{\psi_j(k)\}_{j \in \mathbb{Z}} \in \ell^p(\mathbb{Z}) \), the claim being true for both $p \in \{2, \infty\}$. 
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On can generalize it to other chain graphs, for instance, with varying magnetic field, $A = \{A_j\}_{j \in \mathbb{Z}}$, the ring (half-)perimeters, $\ell = \{\ell_j\}_{j \in \mathbb{Z}}$, etc.
Example: a single flux altered

It is believed that local perturbations give rise to eigenvalues in the gaps. While often true, it need not be the case generally.

We suppose that the field is modified on a single ring, i.e. \( A = \{ \ldots, A_1, A_2, \ldots \} \), the we have a single simple eigenvalue in each gap provided \( |\cos A_1\pi| > 1 \), otherwise the spectrum does not change.

In particular, the perturbation may give rise to no eigenvalues in gaps at all; note that this happens if the perturbed ring is 'further from the non-magnetic case'.

Note also that the eigenvalue may split from the spectral band of the unperturbed system and lies between this band and the nearest eigenvalue of infinite multiplicity. When we change the magnetic field, the eigenvalue may absorbed in the same band. On the other hand no eigenvalue emerges from the degenerate band.
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Can periodic graphs have “wilder” spectra?

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$$u_{n+1} + u_{n-1} + 2\lambda \cos(2\pi(\omega + n\alpha))u_n = \epsilon u_n$$

for $\lambda = 1$, otherwise called Harper equation, as a function of $\alpha$
Nice mathematics, but do such things exist?

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On the physical side, the effect remained theoretical for a long time and thought of in terms of the mentioned setting, with lattice and and a homogeneous field providing the needed two length scales, generically incommensurable, from the lattice spacing and the cyclotron radius.
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Only recently an experimental realization of the original concept was achieved using a graphene lattice [Dean et al’13], [Ponomarenko’13].
A chain with linear field growth

Suppose that $A_j = \alpha j + \theta$ holds for some $\alpha, \theta \in \mathbb{R}$ and every $j \in \mathbb{Z}$. We need a stronger version of duality proved in [Pankrashkin'13] using boundary triples: we exclude $\sigma_D = \{k^2 : k \in \mathbb{N}\}$ and introduce $s(x;z) = \begin{cases} \sin(x\sqrt{z})\sqrt{z} & \text{for } z \neq 0, \\ x & \text{for } z = 0, \end{cases}$ and $c(x;z) = \cos(x\sqrt{z})$.

Theorem (after Pankrashkin'13)

For any interval $J \subset \mathbb{R} \setminus \sigma_D$, the operator $(H_{\gamma}, A_J)$ is unitarily equivalent to the pre-image $\eta(\eta(L_A)\eta(J))$, where $L_A$ is the operator on $\ell^2(\mathbb{Z})$ acting as $(L_Aq\phi_j)_j = 2 \cos(\alpha_j \pi)\phi_{j+1} + 2 \cos(\alpha_{j-1} \pi)\phi_{j-1}$ and $\eta(z) = \gamma s(\pi;z) + 2 c(\pi;z) + 2 s'(\pi;z)$.

Important: a simple gauge transformation shows that it is only the fractional part of $\alpha_j$ which matters. Consequently, the case of a rational slope, $\alpha = p/q$, is reduced to a periodic problem allowing the usual Floquet-Bloch treatment.
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\end{cases}
\]

and

\[
c(x; z) = \cos(x\sqrt{z})
\]

\textbf{Theorem (after Pankrashkin’13)}

For any interval \( J \subset \mathbb{R} \setminus \sigma_D \), the operator \( (H_{\gamma, A})_J \) is unitarily equivalent to the pre-image \( \eta^{-1}((L_A)_{\eta(J)}) \), where \( L_A \) is the operator on \( \ell^2(\mathbb{Z}) \) acting as \( (L_A q \varphi)_j = 2 \cos(A_j \pi) \varphi_{j+1} + 2 \cos(A_{j-1} \pi) \varphi_{j-1} \) and

\[
\eta(z) := \gamma s(\pi; z) + 2c(\pi; z) + 2s'(\pi; z)
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\textbf{Important:} a simple gauge transformation shows that it is only the fractional part of \( A_j \) which matters
A chain with linear field growth

Suppose that $A_j = \alpha j + \theta$ holds for some $\alpha, \theta \in \mathbb{R}$ and every $j \in \mathbb{Z}$

We need a stronger version of duality proved in [Pankrashkin’13] using boundary triples: we exclude $\sigma_D = \{k^2 : k \in \mathbb{N}\}$ and introduce

$$s(x; z) = \begin{cases} \frac{\sin(x\sqrt{z})}{\sqrt{z}} & \text{for } z \neq 0, \\ x & \text{for } z = 0, \end{cases} \quad \text{and} \quad c(x; z) = \cos(x\sqrt{z})$$

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Important: a simple gauge transformation shows that it is only the fractional part of $A_j$ which matters. Consequently, the case of a rational slope, $\alpha = p/q$, is reduced to a periodic problem allowing the usual Floquet-Bloch treatment.
The chain graph spectrum

For *irrational slope* duality allows to transform the problem into Harper equation. In this way we get *in a cheap way* a rather nontrivial result:

Theorem (E-Vášata'17)

Let $A_j = \alpha_j + \theta$ for some $\alpha, \theta \in \mathbb{R}$ and every $j \in \mathbb{Z}$. Then for the spectrum $\sigma(-\Delta, A)$ the following holds:

(a) If $\alpha, \theta \in \mathbb{Z}$ and $\gamma = 0$, then $\sigma_{ac}(-\Delta, A) = [0, \infty)$ and $\sigma_{pp}(-\Delta, A) = \{n^2 | n \in \mathbb{N}\}$

(b) If $\gamma \neq 0$ and $\alpha = p/q$ with $p, q$ relatively prime, $\alpha_j + \theta + \frac{1}{2} \in \mathbb{Z}$ for all $j = 0, \ldots, q - 1$, then $-\Delta, A$ has infinitely degenerate ev's $\{n^2 | n \in \mathbb{N}\}$ interlaced with an ac part consisting of $q$-tuples of closed intervals

(c) If the situation is as in (b) but $\alpha_j + \theta + \frac{1}{2} \in \mathbb{Z}$ holds for some $j = 0, \ldots, q - 1$, then the spectrum $\sigma(-\Delta, A)$ consists of infinitely degenerate eigenvalues only, the Dirichlet ones plus $q$ distinct others in each interval $(-\infty, 1)$ and $(n^2, (n+1)^2)$. P.E.: Spectra of periodic graphs DOGW 2019 Graz February 25, 2019 - 13 -
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Theorem (E-Vašata’17, cont’d)

(d) If $\gamma \neq 0$ and $\alpha \notin \mathbb{Q}$, then $\sigma(-\Delta_{\gamma,A})$ does not depend on $\theta$ and it is a disjoint union of the isolated-point family $\{n^2 | n \in \mathbb{N}\}$ and Cantor sets, one inside each interval $(-\infty, 1)$ and $(n^2, (n + 1)^2)$, $n \in \mathbb{N}$. Moreover, the overall Lebesgue measure of $\sigma(-\Delta_{\gamma,A})$ is zero.
The chain graph spectrum, continued

Theorem (E-Vašata’17, cont’d)

\[(d) \text{ If } \gamma \neq 0 \text{ and } \alpha \notin \mathbb{Q}, \text{ then } \sigma(-\Delta_{\gamma,A}) \text{ does not depend on } \theta \text{ and it is a disjoint union of the isolated-point family } \{n^2| n \in \mathbb{N}\} \text{ and Cantor sets, one inside each interval } (-\infty, 1) \text{ and } (n^2, (n+1)^2), n \in \mathbb{N}. \text{ Moreover, the overall Lebesgue measure of } \sigma(-\Delta_{\gamma,A}) \text{ is zero.}\]

Furthermore, using a result of [Last-Shamis’16] one can also show

Proposition

Let \(A_j = \alpha j + \theta\) for some \(\alpha, \theta \in \mathbb{R}\) and every \(j \in \mathbb{Z}\). There exist a dense \(G_\delta\) set of the slopes \(\alpha\) for which, and all \(\theta\), the Haussdorff dimension

\[\dim_H \sigma(-\Delta_{\gamma,A}) = 0\]
Changing topic: graphs with a few gaps only

The graphs in the previous example had ‘many’ gaps indeed. Let us now ask whether periodic graphs can have ‘just a few’ gaps
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This is the celebrated Bethe–Sommerfeld conjecture to which we have nowadays an affirmative answer in a large number of cases.

In quantum graphs, ‘this is not a strict law’ by [Berkolaiko-Kuchment'13].

For instance, we know that infinitely many gaps can be created by a graph ‘decoration’, cf. [Schenker-Aizenman'00], [Kuchment'04].

The question arises, whether it is a ‘law’ at all?. In other words, do infinite periodic graphs having a finite nonzero number of open gaps exist?

From obvious reasons we would call them Bethe–Sommerfeld graphs.
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The question arises, whether it is a ‘law’ at all?. In other words, do infinite periodic graphs having a *finite nonzero* number of open gaps exist? From obvious reasons we would call them *Bethe–Sommerfeld graphs*.
The answer depends on the vertex coupling. Recall that the standard conditions for self-adjoint coupling of $n$ edges at a vertex,

$$(U - I)\psi + i(U + I)\psi' = 0,$$

where $U$ is an $n \times n$ unitary matrix.
Scale-invariant coupling

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**Theorem ([E-Turek’17])**

An infinite periodic quantum graph \textit{does not belong} to the Bethe-Sommerfeld class if the couplings at its vertices are scale-invariant.
Proof idea

The spectrum is determined by secular equation [BB’15]: we define

\[ F(k; \vec{\vartheta}) := \det \left( I - e^{i(A+kL)}S(k) \right), \]

where the \(2E \times 2E\) matrices \(A\), \(L\), and \(S\) are as follows: the diagonal matrix \(L\) is given by the lengths of the directed edges (bonds) of \(\Gamma\),...
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Then $k^2 \in \sigma(H)$ holds if there is a quasimomentum values $\vec{\vartheta} \in (-\pi, \pi)^\nu$ such that the equation $F(k; \vec{\vartheta}) = 0$ is satisfied.
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We note that $F(k; \vec{\vartheta})$ depends on $\vec{\vartheta}$ and $(k\ell_0, k\ell_1, \ldots, k\ell_d)$, where $\{\ell_0, \ell_1, \ldots, \ell_d\}$, $d + 1 \leq E$ are the mutually different edge lengths of $\Gamma$. 
The spectrum is determined by \textit{secular equation} \cite{BB15}: we define

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We note that \(F(k; \vec{\vartheta})\) depends on \(\vec{\vartheta}\) and \((k\ell_0, k\ell_1, \ldots, k\ell_d)\), where \(\{\ell_0, \ell_1, \ldots, \ell_d\}, \, d + 1 \leq E\) are the mutually different edge lengths of \(\Gamma\). If the \(\ell\)'s are rationally related, the function is \textit{periodic} in \(k\), hence if there is a gap, there are \textit{infinitely many of them}. 

P.E.: Spectra of periodic graphs
Proof idea, and an extension

If the lengths are \textit{not} rationally related, their ratios can be \textit{approximated by rationals} with an arbitrary precision.
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Recall next that the vertex conditions can be equivalently written as

\[
\begin{pmatrix}
I^{(r)} & T \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi'
\end{pmatrix}
=
\begin{pmatrix}
S & 0 \\
-T^* & I^{(n-r)}
\end{pmatrix}
\begin{pmatrix}
\psi
\end{pmatrix}
\]

for certain \( r \), \( S \), and \( T \), where \( I^{(r)} \) is the identity matrix of order \( r \); the coupling is scale-invariant if and only if the square matrix \( S = 0 \)
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for certain $r$, $S$, and $T$, where $I^{(r)}$ is the identity matrix of order $r$; the coupling is scale-invariant if and only if the square matrix $S = 0$

We will consider two associated quantum graph Hamiltonians, $H$ with the above vertex coupling, and $H_0$ where we replace $S$ by zero
A result for this associated pair

Proposition ([E-Turek’17])

For the spectra $\sigma(H)$ and $\sigma(H_0)$ the following claims hold true:

(i) If $\sigma(H_0)$ has an open gap, then $\sigma(H)$ has infinitely many gaps.

(ii) If the edge lengths are rationally dependent, then the gaps of $\sigma(H)$ asymptotically coincide with those of $\sigma(H_0)$. 

Proof idea:
The argument is based on the following observation: the on-shell S-matrix for $S(k) = -I(n) + 2(I(r)T^* - 1)(I(r) + TT^* - 1) - 1(I(r)T)$.
Hence the scale-invariant part is, naturally, independent of $k$, and the Robin part is $O(k - 1)$. The same is true for $S(k)$, and as consequence, the spectrum at high energies is mostly determined by the scale-invariant part. □
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$$S(k) = -I^{(n)} + 2 \begin{pmatrix} I^{(r)} & \end{pmatrix} \begin{pmatrix} I^{(r)} + TT^* - \frac{1}{ik} S \end{pmatrix}^{-1} \begin{pmatrix} I^{(r)} & T \end{pmatrix}$$

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So, are there any BS graphs?

Theorem (E-Turek'17)

Bethe–Sommerfeld graphs exist.

As usual with existence claims, it is enough to demonstrate an example. With this aim we are going to revisit the model of a rectangular lattice graph with $\delta$-coupling introduced in [E'96, E-Gawlista'96].
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We can give an affirmative answer to this question:

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Spectral condition

According to [E’96], a number $k^2 > 0$ belongs to a gap if and only if $k > 0$ satisfies the gap condition, which reads

$$\tan \left( \frac{ka}{2} - \frac{\pi}{2} \left\lfloor \frac{ka}{\pi} \right\rfloor \right) + \tan \left( \frac{kb}{2} - \frac{\pi}{2} \left\lfloor \frac{kb}{\pi} \right\rfloor \right) < \frac{\alpha}{2k} \quad \text{for} \ \alpha > 0$$

and

$$\cot \left( \frac{ka}{2} - \frac{\pi}{2} \left\lfloor \frac{ka}{\pi} \right\rfloor \right) + \cot \left( \frac{kb}{2} - \frac{\pi}{2} \left\lfloor \frac{kb}{\pi} \right\rfloor \right) < \frac{|\alpha|}{2k} \quad \text{for} \ \alpha < 0,$$

where we denote the edge lengths $\ell_j, j = 1, 2$, as $a, b$; we neglect the Kirchhoff case, $\alpha = 0$, where $\sigma(H) = [0, \infty)$. 
Spectral condition

According to [E’96], a number $k^2 > 0$ belongs to a gap if and only if $k > 0$ satisfies the gap condition, which reads

$$\tan \left( \frac{ka}{2} - \frac{\pi}{2} \left\lfloor \frac{ka}{\pi} \right\rfloor \right) + \tan \left( \frac{kb}{2} - \frac{\pi}{2} \left\lfloor \frac{kb}{\pi} \right\rfloor \right) < \frac{\alpha}{2k} \quad \text{for } \alpha > 0$$

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where we denote the edge lengths $\ell_j, j = 1, 2$, as $a, b$; we neglect the Kirchhoff case, $\alpha = 0$, where $\sigma(H) = [0, \infty)$.

Note that for $\alpha < 0$ the spectrum extends to the negative part of the real axis and may have a gap there, which is not important here because there is not more than a single negative gap, and this gap always extends to positive values.
What is known

The spectrum depends on the ratio $\theta = \frac{\ell_1}{\ell_2}$. If $\theta$ is rational, $\sigma(H)$ has infinitely many gaps unless $\alpha = 0$ in which case $\sigma(H) = [0, \infty)$. The same is true if $\theta$ is an irrational well approximable by rationals, which means equivalently that in the continuous fraction representation $\theta = [a_0; a_1, a_2, ...]$ the sequence $\{a_j\}$ is unbounded.

On the other hand, $\theta \in \mathbb{R}$ is badly approximable if there is a $c > 0$ such that $\left|\theta - \frac{p}{q}\right| > \frac{c}{q^2}$ for all $p, q \in \mathbb{Z}$ with $q \neq 0$. For such numbers we define the Markov constant by $\mu(\theta) := \inf\{c > 0 \mid \exists \infty (p, q) \in \mathbb{N}^2 (\left|\theta - \frac{p}{q}\right| < \frac{c}{q^2})\}$ (we note that $\mu(\theta) = \mu(\theta - 1)$).
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(we note that $\mu(\theta) = \mu(\theta^{-1})$) and its ‘one-sided analogues’.
The golden mean situation

For example, consider the golden mean, $\phi = \frac{\sqrt{5}+1}{2} = [1; 1, 1, \ldots]$, which can be regarded as the ‘worst’ irrational.
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The answer is not a priori clear: let us plot the minima of the function appearing in the first gap condition, i.e. for \( \alpha > 0 \)

Note that they approach the limit values \textit{from above}, also that the series open at \( \frac{\pi^2}{\sqrt{5ab}} \phi^{\mp 1/2} |n^2 - m^2 - nm|, \ n, m \in \mathbb{N} \) \[E-Gawlista'96\]
But a closer look shows a more complex picture

**Theorem ([E-Turek’17])**

Let \( \frac{a}{b} = \phi = \frac{\sqrt{5}+1}{2} \), then the following claims are valid:

(i) If \( \alpha > \frac{\pi^2}{\sqrt{5}a} \) or \( \alpha \leq -\frac{\pi^2}{\sqrt{5}a} \), there are **infinitely many spectral gaps**.

(ii) If
\[
-\frac{2\pi}{a} \tan \left( \frac{3 - \sqrt{5}}{4} \pi \right) \leq \alpha \leq \frac{\pi^2}{\sqrt{5}a},
\]
there are **no gaps** in the positive spectrum.

(iii) If
\[
-\frac{\pi^2}{\sqrt{5}a} < \alpha < -\frac{2\pi}{a} \tan \left( \frac{3 - \sqrt{5}}{4} \pi \right),
\]
there is a **nonzero and finite number of gaps** in the positive spectrum.
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Let $\frac{a}{b} = \phi = \frac{\sqrt{5} + 1}{2}$, then the following claims are valid:

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(iii) If

$$-\frac{\pi^2}{\sqrt{5}a} < \alpha < -\frac{2\pi}{a} \tan \left( \frac{3 - \sqrt{5}}{4} \pi \right),$$

there is a nonzero and finite number of gaps in the positive spectrum.

Corollary

The above theorem about the existence of BS graphs is valid.
More about this example

The window in which the golden-mean lattice has the Bethe–Sommerfeld property is narrow, it is roughly $4.298 \lesssim -\alpha a \lesssim 4.414$. 
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We are also able to control the number of gaps in the BS regime:

**Theorem ([E-Turek’17])**

For a given $N \in \mathbb{N}$, there are exactly $N$ gaps in the positive spectrum if and only if $\alpha$ is chosen within the bounds

$$- \frac{2\pi}{\sqrt{5}a} \left( \frac{\phi^{2(N+1)} - \phi^{-2(N+1)}}{\sqrt{5}a} \right) \tan \left( \frac{\pi}{2} \phi^{-2(N+1)} \right) \leq \alpha < - \frac{2\pi}{\sqrt{5}a} \left( \frac{\phi^{2N} - \phi^{-2N}}{\sqrt{5}a} \right) \tan \left( \frac{\pi}{2} \phi^{-2N} \right).$$
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Note that the numbers $A_j := \frac{2\pi (\phi^{2j} - \phi^{-2j})}{\sqrt{5}} \tan \left( \frac{\pi}{2} \phi^{-2j} \right)$ form an increasing sequence the first element of which is $A_1 = 2\pi \tan \left( \frac{3-\sqrt{5}}{4} \pi \right)$ and $A_j < \frac{\pi^2}{\sqrt{5}}$ for all $j \in \mathbb{N}$. 
More general result

Proofs of the above results are based on properties of Diophantine approximations. In a similar way one can prove
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**Theorem ([E-Turek’17])**

Let \( \theta = \frac{a}{b} \) and define

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\gamma_+ := \min \left\{ \inf_{m \in \mathbb{N}} \left\{ \frac{2m \pi}{a} \tan \left( \frac{\pi}{2} \left( m\theta^{-1} - \lfloor m\theta^{-1} \rfloor \right) \right) \right\}, \inf_{m \in \mathbb{N}} \left\{ \frac{2m \pi}{b} \tan \left( \frac{\pi}{2} \left( m\theta - \lfloor m\theta \rfloor \right) \right) \right\} \right\}
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and \( \gamma_- \) similarly with \( \lfloor \cdot \rfloor \) replaced by \( \lceil \cdot \rceil \).
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and \( \gamma_- \) similarly with \( \lfloor \cdot \rfloor \) replaced by \( \lceil \cdot \rceil \). If the coupling constant \( \alpha \) satisfies

\[
\gamma_\pm < \pm \alpha < \frac{\pi^2}{\max\{a, b\}} \mu(\theta),
\]

then there is a nonzero and finite number of gaps in the positive spectrum.
Another application of periodic quantum graphs

Square lattice graphs were recently suggested a tool to model the anomalous Hall effect, cf. [Středa-Kučera’15], i.e. the situation when a Hall voltage appears without the presence of an external magnetic field.
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This motivated us to investigate situations where such an inherent rotation can take place, not at the graphs edges but in its vertices.
The coupling choice

In the vertex coupling conditions mentioned above,

\[(U - I)\psi + i(U + I)\psi' = 0,\]

we choose the unitary matrix \(U\) of the form

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & 1
\end{bmatrix};
\]

the aim is to achieve 'maximum rotation' at a fixed energy, conventionally corresponding the momentum \(k = 1\). Recall that one has \(U = S(1)\)

Componentwise, with \(\psi_j = \psi_j(0^+)\) and \(\psi'_j = \psi'_j(0^+)\), we have

\[(\psi_{j+1} - \psi_j) + i(\psi'_{j+1} + \psi'_j) = 0,\]

\(j \in \mathbb{Z}(\text{mod} N)\), which is non-trivial only for \(N \geq 3\) and obviously non-invariant w.r.t. the reverse in the edge numbering order.
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Spectrum and on-shell S-matrix

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$$\kappa = \tan \frac{\pi m}{N}$$

with $m$ running through $1, \ldots, \left[\frac{N}{2}\right]$ for $N$ odd and $1, \ldots, \left[\frac{N-1}{2}\right]$ for $N$ even. Thus $\sigma_{\text{disc}}(H)$ is always nonempty.
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For any vertex coupling $U$ the S-matrix at the momentum $k$ is

$$S(k) = \frac{k - 1 + (k + 1)U}{k + 1 + (k - 1)U};$$

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It might seem that the transport becomes trivial at small and high energies, since $\lim_{k \to 0} S(k) = -I$ and $\lim_{k \to \infty} S(k) = I$. 
The on-shell S-matrix, continued

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Denoting $\eta := \frac{1-k}{1+k}$ we get by a straightforward computation

$$S_{ij}(k) = \frac{1 - \eta^2}{1 - \eta^N} \left\{ -\eta \frac{1 - \eta^{N-2}}{1 - \eta^2} \delta_{ij} + (1 - \delta_{ij}) \eta^{(j-i-1)(\text{mod} N)} \right\}$$
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This suggests, in particular, that the high-energy behavior, \( \eta \to -1 \), could be determined by the \textit{parity} of the vertex degree \( N \)
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This suggests, in particular, that the high-energy behavior, $\eta \to -1-$, could be determined by the parity of the vertex degree $N$. Indeed, we see that $\lim_{k \to \infty} S(k) = I$ holds for $N = 3$ and more generally for all odd $N$. 

P.E.: Spectra of periodic graphs

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The on-shell $S$-matrix, continued

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On the hand, for $N = 4$ we have

$$S(k) = \frac{1}{1 + \eta^2} \begin{pmatrix} -\eta & \eta^2 & \eta & 1 \\ \eta^2 & -\eta & 1 & \eta \\ \eta & \eta^2 & -\eta & 1 \\ 1 & \eta & \eta^2 & -\eta \end{pmatrix}$$

so the limit is not a multiple of identity.
The on-shell S-matrix, continued

However, more caution is needed; the formal limits is false if $+1$ or $-1$ are eigenvalues of $U$. A counterexample is provided by Kirchhoff coupling where $U$ has only $\pm 1$ as its eigenvalues; the corresponding S-matrix is $k$-independent and not a multiple of the identity.

Denoting $\eta := \frac{1-k}{1+k}$ we get by a straightforward computation

$$S_{ij}(k) = \frac{1 - \eta^2}{1 - \eta^N} \left\{ -\eta \frac{1 - \eta^{N-2}}{1 - \eta^2} \delta_{ij} + (1 - \delta_{ij}) \eta^{(j-i-1)(\text{mod } N)} \right\}$$

This suggests, in particular, that the high-energy behavior, $\eta \to -1$, could be determined by the parity of the vertex degree $N$. Indeed, we see that $\lim_{k \to \infty} S(k) = \mathbf{I}$ holds for $N = 3$ and more generally for all odd $N$.

On the hand, for $N = 4$ we have

$$S(k) = \frac{1}{1 + \eta^2} \begin{pmatrix}
-\eta & 1 & \eta & \eta^2 \\
\eta^2 & -\eta & 1 & \eta \\
\eta & \eta^2 & -\eta & 1 \\
1 & \eta & \eta^2 & -\eta \\
\end{pmatrix}$$

so the limit is not a multiple of identity; the same is true for any even $N$.
Comparison of two lattices

\[ (\theta_1 + \theta_2) k \sin k \ell - (k^2 - 1)(\cos \theta_1 + \cos \theta_2) + 2(k^2 + 1) \cos k \ell = 0 \]

and

\[ -i (\theta_1 + \theta_2) k^2 \sin k \ell (3 + 6k^2 - k^4 + 4d \theta (k^2 - 1) + (k^2 + 3)^2 \cos 2k \ell) = 0, \]

where

\[ d \theta := \cos \theta_1 + \cos(\theta_1 - \theta_2) + \cos \theta_2, \]

\( \ell (\theta_1, \theta_2) \in [-\pi \ell, \pi \ell] \)

2 is the quasimomentum, but tedious to solve except the flat band cases, \( \sin k \ell = 0 \). However, we can present the band solution in a graphical form.
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Spectral condition for the cases are easy to derive,

\[ (\theta_1 + \theta_2) k \sin k \ell \left[ (k^2 - 1)(\cos \theta_1 + \cos \theta_2) + 2(k^2 + 1) \cos k \ell \right] = 0 \]

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Other interesting results can be found, say, concerning interpolation between the $\delta$-coupling and the present one. This is our next topic.
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Other interesting results can be found, say, concerning *interpolation* between the \( \delta \)-coupling and the present one. This is our next topic.
More generally on vertex coupling symmetries

A symmetry is described by an invertible map in the space of boundary values, $\Theta : \mathbb{C}^n \to \mathbb{C}^n$. A vertex coupling is symmetric w.r.t $\Theta$ if the usual matching condition is equivalent to

$$(U - I)\Theta \psi(0) + i(U + I)\Theta \psi'(0) = 0,$$

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- **Mirror symmetric** couplings have the matching conditions invariant against \( (\psi_1(0), \ldots, \psi_n(0)) \mapsto (\psi_n(0), \ldots, \psi_1(0)) \) and the same for the derivatives; in other words, \( \Theta = A \), the matrix with the entries equal to 1 on the main antidiagonal and zero otherwise.
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- The subset of \textit{permutation-invariant} couplings is a two-parameter family, $U = aI + bJ$, where $J$ denotes the matrix with all the entries equal to one; the parameters satisfy $|a| = 1$ and $|a + nb| = 1$. 

\textit{Time-reversal-invariant} couplings: in this case $\Theta$ is replaced by the antilinear operator of complex conjugation, and using relations $U^t \bar{U} = \bar{U}U^t = I$ we find easily that the matrix describing the coupling must be now invariant w.r.t. transposition, $U = U^t$.
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Symmetries, continued

Our focus here is on rotationally symmetric vertex coupling, i.e. those independent of cyclic permutations of the entries of $\Psi(0)$ and $\Psi'(0)$. They correspond to

$$\Theta = R := \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
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**Proposition**

A *rotationally symmetric* vertex coupling is *mirror symmetric* if and only if it is *time-reversal-invariant*. 
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**Proposition**

A *rotationally symmetric vertex coupling is mirror symmetric if and only if it is time-reversal-invariant.*

Note also that while the notions of permutation-invariant and time-reversal-invariant couplings are universal in the sense that they do *not* require embedding in an ambient space, the other two mentioned above make sense only if we think of the graph $\Gamma$ as of *an embedded object*
Circulant matrices

Matrices satisfying $\Theta^{-1}U\Theta = U$ for the above $\Theta = R$ are circulant matrices, which generally take the form

$$
\begin{pmatrix}
  c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\
  c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & c_0 & \vdots \\
  c_2 & \vdots & \ddots & \ddots & c_1 \\
  c_1 & c_2 & \cdots & c_{n-1} & c_0
\end{pmatrix},
$$

being a particular case of Toeplitz matrices. The first row, i.e. the vector $c = (c_0, c_1, \ldots, c_{n-1})$, is called generator of $C$. 

P.E.: Spectra of periodic graphs

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Let us recall basic properties of such matrices. The vectors

\[
v_k = \frac{1}{\sqrt{n}} \left(1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k}\right)^T, \quad k = 0, 1, \ldots, n-1,
\]

with $\omega := e^{2\pi i/n}$ are its normalized eigenvectors corresponding to

\[
\lambda_k = c_0 + c_1\omega^k + c_2\omega^{2k} + \cdots + c_{n-1}\omega^{(n-1)k}, \quad k = 0, 1, \ldots, n-1.
\]
Circulant matrices, continued

Every circulant matrix $C$ is diagonalized by the Discrete Fourier Transform matrix

$$F = \begin{pmatrix}
1 & 1 & 1 & 1 & \ldots & 1 \\
1 & \omega & \omega^2 & \omega^3 & \ldots & \omega^{n-1} \\
1 & \omega^2 & \omega^4 & \omega^6 & \ldots & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \ldots & \omega^{(n-1)^2}
\end{pmatrix},$$

in other words, $D = C F^{-1} F$ is a diagonal matrix with $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ on the diagonal. Since $F^{-1} = \frac{1}{n} F^*$, we can express from this relation $c_j = \frac{1}{n} (\lambda_0 + \lambda_1 \omega^{-j} + \lambda_2 \omega^{-2j} + \cdots + \lambda_{n-1} \omega^{-(n-1)j})$.

A circulant matrix is unitary if and only if $|\lambda_j| = 1$ holds for all indices $j = 0, 1, \ldots, n-1$. Consequently, circulant unitary matrices of order $n$ are parametrized by an $n$-tuple of real numbers.
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A class of interpolating couplings

Our aim is now to investigate a class of couplings interpolating between the ‘maximally rotational’ one discussed above and the usual $\delta$ coupling referring to

$$U = -I + \frac{2}{n + i\alpha} J,$$

where $\alpha \in \mathbb{R}$ is a parameter; the case $\alpha = 0$ is called *Kirchhoff coupling*. 

We will construct them using their eigenvalues. For the circulant matrix $U(1)$ we get

$$\lambda_k = \omega_k$$

if $k = 0, 1, \ldots, n-1$. 

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We seek a family of unitary matrices $\{U(t) : t \in [0, 1]\}$ such that

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\begin{align*}
    U(0) &= -I + \frac{2}{n + i\alpha} J \quad \text{and} \quad U(1) = R; \\
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    U(t) \text{ is unitary circulant for all } t \in [0, 1].
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$$\lambda_k = \omega^k \quad \text{if } k = 0, 1, \ldots, n-1.$$
A class of interpolating couplings

Similarly, the eigenvalues of $U(0)$ are

$$\lambda_k = -1 + \frac{2}{n + i\alpha} \sum_{j=0}^{n-1} \omega_{jk} = \begin{cases} \frac{n-i\alpha}{n+i\alpha} & \text{for } k = 0; \\ -1 & \text{for } k \geq 1. \end{cases}$$

For brevity, let us set $\frac{n-i\alpha}{n+i\alpha} = e^{-i\gamma}$; note that

$$\gamma \begin{cases} = 0 & \text{for } \alpha = 0; \\ \in (0, \pi) & \text{for } \alpha > 0; \\ \in (-\pi, 0) & \text{for } \alpha < 0. \end{cases}$$
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Then we are looking for a matrix with the eigenvalues

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\lambda_k(t) = \begin{cases} 
e^{-i(1-t)\gamma} & \text{for } k = 0; \\
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\end{cases}
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for all $t \in [0, 1]$ which would satisfy the stated requirements.
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for all $t \in [0, 1]$ which would satisfy the stated requirements corresponding to

$$c_j(t) = \frac{1}{n} \left( e^{-i(1-t)\gamma} - \sum_{k=1}^{n-1} e^{i\pi t \left(\frac{2k}{n} - 1\right)} \cdot \omega^{-kj} \right)$$
Spectrum of a star graph

Let $\Gamma$ be a star with $n$ semi-infinite edges, then it is easy to check that the essential/continuous spectrum is $[0, \infty)$. The negative spectrum is nonempty for any $t \in (0, 1]$ and $n \geq 3$. Indeed, writing $\psi_j(x) = b_j e^{-\kappa x}$, we get the spectral condition

$$\det[(U(t) - I) - i\kappa (U(t) + I)] = 0,$$

which is equivalent to

$$\kappa = -i\lambda_j(t) - 1\lambda_j(t) + 1$$

for some $0 \leq j \leq n - 1$. This yields

$$\kappa = -\tan(1 - t)\gamma/2$$

for $\alpha < 0$ and $t \neq 1$

$$\kappa = -\cot^{2}(j(n - 1)/2)\pi t$$

for $n \geq 3$ and $j < \lfloor n/2 \rfloor$. These solutions give rise to the eigenvalues $-\kappa^2 < 0$ of the star graph: there is a negative eigenvalue $-\tan^2(1 - t)\gamma^2$ whenever $\alpha < 0$; if $n \geq 3$, there is an additional \(\lfloor n - 1/2 \rfloor\)-tuple for each $t \in (0, 1]$, namely $-\cot^{2}(j(n - 1)/2)\pi t$ with $j$ running through $1, \ldots, n - 1/2$ for $n$ odd and $1, \ldots, n/2 - 1$ for $n$ even.
Spectrum of a star graph

Let $\Gamma$ be star with $n$ semi-infinite edges, then it is easy to check that the essential/continuous spectrum is $[0, \infty)$. The negative spectrum is nonempty for any $t \in (0, 1]$ and $n \geq 3$. Indeed, writing $\psi_j(x) = b_j e^{-\kappa x}$, we get the spectral condition

$$\det[(U(t) - I) - i\kappa(U(t) + I)] = 0,$$

which is equivalent to $\kappa = -i \frac{\lambda_j(t)-1}{\lambda_j(t)+1}$ for some $0 \leq j \leq n - 1$. 

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$$\kappa = -\tan \left(\frac{1 - t}{2}\right) \gamma$$

for $\alpha < 0$ and $t \neq 1$

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Eigenvalue limiting behavior

Finally, look at what the eigenvalues do in the limits $t \to 0^+$ (δ coupling):
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If $t \to 0^+$, all the eigenvalues $-\kappa^2$ diverge to $-\infty$ except for $-\tan^2(1-t)\gamma^2$ occurring for $\alpha < 0$, which approaches the value $-\tan^2(\gamma/2) = -\alpha^2/n^2$. When $t = 0$, the ‘additional’ eigenvalues thus disappear and the system has only one simple negative eigenvalue $-\alpha^2/n^2$ for $\alpha < 0$, while for $\alpha \geq 0$ its negative spectrum is empty.

If $t \to 1^-$, the eigenvalues $-\kappa^2$ approach zero and $-\tan^2j/n$, respectively. When $t = 1$, the only negative eigenvalues are $-\tan^2j/n$ with $j$ taking values $1,\ldots,n-1$ for $n$ odd and $1,\ldots,n^2-1$ for $n$ even – as we have seen before.
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Knowing $U(t)$ we easily find the on-shell S-matrix. In particular, its eigenvalues are

$$\mu_j(t) = \frac{k - 1 + (k + 1)\lambda_j(t)}{k + 1 + (k - 1)\lambda_j(t)}$$

for $j = 0, 1, \ldots, n - 1$. 

We are again interested in the high-energy behavior of the S-matrix:

If $n$ is odd, then $\lim_{k \to \infty} \mu_j = 1$ for all $j = 0, 1, \ldots, n - 1$, hence $\lim_{k \to \infty} S(k) = I$.

If $n$ is even, then $\lim_{k \to \infty} \mu_j = 1$ for $j \neq \frac{n}{2}$, while $\lim_{k \to \infty} \mu_{n/2} = -1$, and consequently, the generator of $\lim_{k \to \infty} S(k)$ equals $(1 - 2n, 2n, -2n, 2n, \ldots, -2n, 2n)$. 

As in the 'extreme' rotational case, the S-matrix behaves at high energies differently for odd and even $n$. 

P.E.: Spectra of periodic graphs

DOGW 2019 Graz

February 25, 2019
Scattering

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  $$\left(1 - \frac{2}{n}, \frac{2}{n}, -\frac{2}{n}, \frac{2}{n}, \ldots, -\frac{2}{n}, \frac{2}{n}\right)$$

As in the ‘extreme’ rotational case, the S-matrix behaves at high energies differently for odd and even $n$. 
Square lattice

We consider again a lattice of spacing \( \ell \), now with the interpolating coupling. The treatment is similar to the particular case analyzed earlier; it yields the spectral condition

\[
512 e^{i(\theta_1 + \theta_2)} e^{-i \frac{(1-t)\gamma}{2}} \left[ V_3 k^3 + V_2 k^2 + V_1 k + V_0 \right] = 0 ,
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where

$$V_3 = -\cos \frac{(1-t)\gamma}{2} \sin^2 \frac{\pi t}{4} \sin k\ell(\cos \theta_1 + \cos \theta_2 + 2 \cos k\ell);$$

$$V_2 = 2\sin \frac{(1-t)\gamma}{2} \sin^2 \frac{\pi t}{4} (\cos \theta_1 + \cos k\ell)(\cos \theta_2 + \cos k\ell);$$

$$V_1 = \cos \frac{(1-t)\gamma}{2} \cos^2 \frac{\pi t}{4} \sin k\ell(\cos \theta_1 + \cos \theta_2 - 2 \cos k\ell);$$

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$$V_0 = -2 \sin \left( \frac{1-t}{2} \gamma \right) \cos^2 \frac{\pi t}{4} \sin^2 k\ell.$$

Hence $k^2$ belongs to the spectrum iff there are $\theta_1, \theta_2 \in [-\pi, \pi)$ such that

$$V_3 k^3 + V_2 k^2 + V_1 k + V_0 = 0.$$
The case $\alpha = 0$

If $\alpha = 0$ we have $V_2 = V_0 = 0$ and the spectral condition reduces to

$$\sin k\ell \left[ \left( k^2 \sin^2 \frac{\pi t}{4} - \cos^2 \frac{\pi t}{4} \right) (\cos \theta_1 + \cos \theta_2) - 2 \left( k^2 \sin^2 \frac{\pi t}{4} + \cos^2 \frac{\pi t}{4} \right) \cos k\ell \right] = 0.$$
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It has two types of solutions. The first are infinitely degenerate eigenvalues, squares of

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It has two types of solutions. The first are infinitely degenerate eigenvalues, squares of

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The condition describing spectral bands can rewritten as

\[
\left| k^2 \sin^2 \frac{\pi t}{4} - \cos^2 \frac{\pi t}{4} \right| \geq \left( k^2 \sin^2 \frac{\pi t}{4} + \cos^2 \frac{\pi t}{4} \right) |\cos k\ell|.
\]

Note that if \( t = 0 \) (i.e., pure Kirchhoff), the condition simplifies to \( |\cos k\ell| \leq 1 \), which is satisfied for all \( k \geq 0 \). Putting this trivial case aside, from now on we assume that \( t \in (0, 1] \).
The case $\alpha = 0$, continued

For $t \in (0, 1]$ the spectrum obviously contains infinitely many gaps located in the vicinity of $k = \frac{m\pi}{\ell}$ for $m \in \mathbb{N}$. The band condition for $k > 0$ can be rewritten as

$$\left( k \left| \tan \frac{k\ell}{2} \right| - \cot \frac{\pi t}{4} \right) \left( k \left| \cot \frac{k\ell}{2} \right| - \cot \frac{\pi t}{4} \right) \geq 0$$

and the zeros of the factors correspond to the band edges.
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and the zeros of the factors correspond to the band edges. The latter intersect at points where $k \left| \tan \frac{k\ell}{2} \right| = k \left| \cot \frac{k\ell}{2} \right| = \cot \frac{\pi t}{4}$, i.e., at $(t, k)$ such that

$$k = \left( m - \frac{1}{2} \right) \frac{\pi}{\ell} \quad \text{and} \quad t = \frac{4}{\pi} \arccot \left( m - \frac{1}{2} \right) \frac{\pi}{\ell}, \quad m \in \mathbb{N}.$$ 

At those values of $t$ the spectrum shrinks into a ‘flat band’
The case $\alpha = 0$, continued

**Figure:** Interpolation with the Kirchhoff coupling for $\ell = 1$, the spectrum is indicated by the shaded regions.
The case $\alpha = 0$, continued

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Let us look at the high-energy behavior of gaps. The condition reads

$$\left| \left( 1 + \frac{2}{k^2} \cot^2 \frac{\pi t}{4} + O(k^{-4}) \right) \cos k\ell \right| > 1 \quad \text{as} \quad k \to \infty$$

so it can be satisfied only for $k$ located in small neighborhoods of $\frac{m\pi}{\ell}$. The approximate width of the $m$-th spectral gap is equal to

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in the momentum variable, hence in energy the gaps are asymptotically constant, of the widths $\frac{8}{\ell} \cot \frac{\pi t}{4} + O(m^{-1})$.
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in the momentum variable, hence in energy the gaps are asymptotically constant, of the widths $\frac{8}{\ell} \cot \frac{\pi t}{4} + O(m^{-1})$. Note that the gap width increases as $t$ diminishes from this extreme value but eventually it starts to decrease again, to the point of vanishing at $t = 0$. 
The general case, \( \alpha \in \mathbb{R} \)

The analysis follows the same line but becomes more complicated; the spectral condition now yields spectra, as functions of \( t \), in the form of a *union of pairs of regions*
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Let us summarize our observations about the band spectra of our lattices:

(i) A ‘discontinuity’ at $t = 0$: There is always a spectral band which becomes narrow and strongly negative as $t \to 0$ and eventually disappears. This obviously corresponds to the behavior of the ‘additional’ eigenvalues of a single vertex.
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(ii) **Point degeneracies for $\alpha = 0$:** In the Kirchhoff case spectral bands may collapse to a point at particular values of $t$. It is not the case if $\alpha \neq 0$ where the spectral condition has other solutions which smear these Kirchhoff degenerate eigenvalues into bands of nonzero width, more pronounced as we are going farther from the Kirchhoff case.
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(iii) **Non-monotonicity of the gap widths:** As mentioned already the widths of the gaps are not monotonous with respect to the interpolation parameter, the same is true if $\alpha \neq 0$. 
Illustration, weakly attractive interaction

Figure: Interpolation with an attractive $\delta$ coupling, $\alpha = -4(\sqrt{2} - 1)$, $\ell = 1$. 
Figure: Interpolation with an attractive $\delta$ coupling, $\alpha = -4(\sqrt{2} + 1), \ell = 1$. 
Other properties

(iv) \textit{\( \alpha \)-independence of some bands:} Some curves marking the band edges are independent of \( \alpha \). However, for \( \alpha \neq 0 \) the ‘band edges’ coincide only in parts of the interval \([0, 1]\), as there is a neighborhood of \( t = \frac{4}{\pi} \arccot \frac{3\pi}{2} \) in which there is an additional spectrum.
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(v)\textit{‘Band edge’ regularity:} The curves delineating the band edges are described by analytic functions. In the Kirchhoff case the analyticity is violated only at the points when the curves are crossing, on the other hand, in the case \(\alpha \neq 0\) where the spectrum is a union of bands there are other points where each particular edge is not smooth; needless to say, it remains Lipshitz.
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(vi) **Flat band spreading:** Infinitely degenerate eigenvalues for $t = 1$ may smear when the interpolation parameter decreases, but we know that they also remain in the spectrum. A possible explanation may come from the presence of different ‘elementary’ eigenfunctions in the case $t = 1$, namely ‘Dirichlet’ and ‘Neumann’ type, which could behave differently with respect to $t$. 

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P.E.: Spectra of periodic graphs

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Illustration, larger lattice spacing

Figure: Interpolation with an attractive $\delta$ coupling, $\alpha = -4(\sqrt{2} + 1)$ and $l = 2\pi$. 
And finally, band edges positions

Looking for extrema of the dispersion functions, people usually seek them and the border of the respective Brillouin zone.
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Subsequently, in [E-Kuchment-Winn’10] it was shown that same may be true even for graphs periodic in one dimension

![Diagram of a periodic quantum graph with band edges at \( \Gamma_j \) and \( \Gamma_{j+1} \).]

The number of connecting edges had to be \( N \geq 2 \)
And finally, band edges positions

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The number of connecting edges had to be \( N \geq 2 \). An example:
Band edges, continued

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![Comb-shaped graph](image_url)
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\[
\begin{array}{cccccccc}
| & | & | & | & | & | & | & | \\
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\end{array}
\]

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Its analysis shows:

- **two-sided comb** is transport-friendly, bands dominate.
- **one-sided comb** is transport-unfriendly, gaps dominate.
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Band edges, continued

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- and what about the dispersion curves?
Two-sided comb: dispersion curves
The sources

The examples discussed in this talk come from
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in combination the other papers mentioned in the course of the presentation.
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Thank you for your attention!