Spectra of periodic quantum graphs: more than one would expect

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Only wimps specialize in the general case. Real scientists pursue examples.

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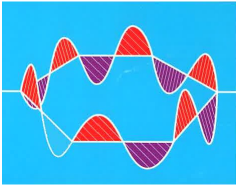

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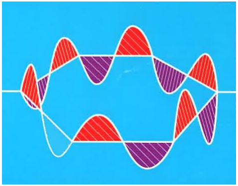

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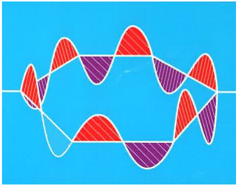

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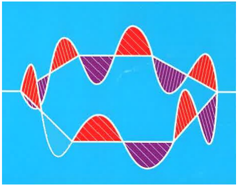

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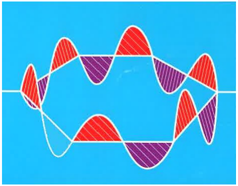

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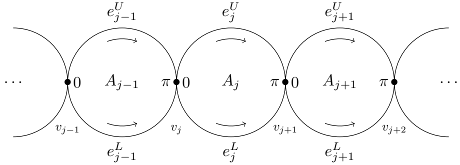
The magnetic chain

To be specific, the chain graph will look as follows



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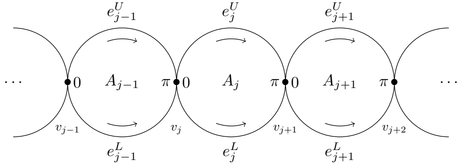

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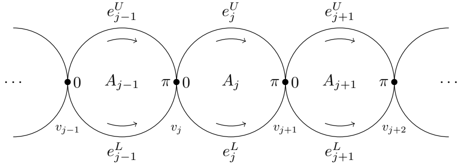

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$$\psi_i(0) = \psi_j(0) =: \psi(0)\,, \quad i,j \in \mathfrak{n}\,, \quad \sum_{i=1}^{n} \mathcal{D}\psi_i(0) = \alpha\,\psi(0)\,,$$

where $\mathfrak{n} = \{1,2,\dots,n\}$ is the index set numbering the edges – in our case $n = 4$ – and $\alpha \in \mathbb{R}$ is the coupling constant

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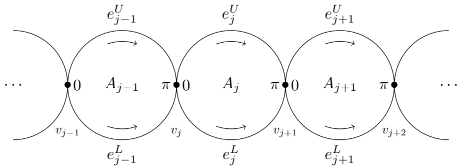

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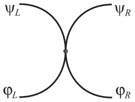
This is a particular case of the general conditions that make the operator self-adjoint [Kostrykin-Schrader'03]

Floquet-Bloch analysis of the fully periodic case



We write $\psi_L(x) = \mathrm{e}^{-iAx}(C_L^+\mathrm{e}^{ikx} + C_L^-\mathrm{e}^{-ikx})$ for $x \in [-\pi/2, 0]$ and energy $E := k^2 \neq 0$, and similarly for the other three components

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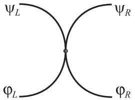

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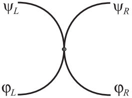

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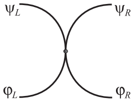

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$$\sin k\pi \cos A\pi(\mathrm{e}^{2i\theta} - 2\xi(k)\mathrm{e}^{i\theta} + 1) = 0$$

with

$$\xi(k) := \frac{1}{\cos A\pi}\left(\cos k\pi + \frac{\alpha}{4k}\sin k\pi\right),$$

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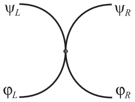

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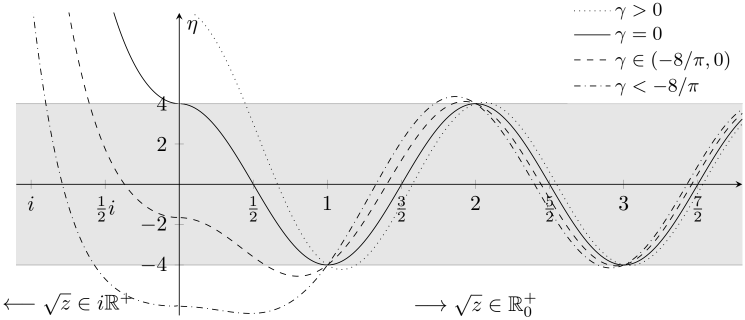
Apart from the cases $A - \frac{1}{2} \in \mathbb{Z}$ and $k \in \mathbb{N}$ we have $k^2 \in \sigma(-\Delta_\alpha)$ *iff* the condition $|\xi(k)| \leq 1$ is satisfied.

In picture: determining the spectral bands



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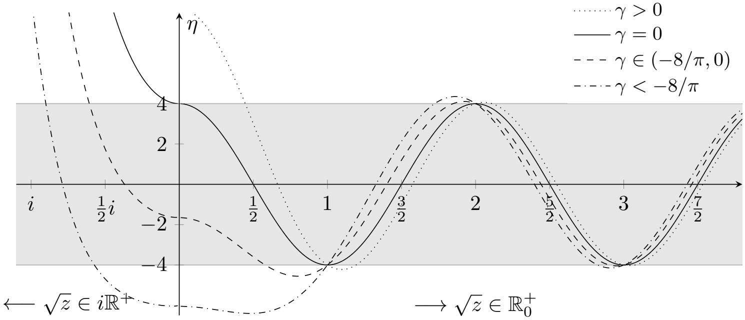

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For $A - \frac{1}{2} \notin \mathbb{Z}$ the situation is similar, just the width of the band changes to $4 \cos A\pi$, on the other hand, for $A - \frac{1}{2} \in \mathbb{Z}$ it *shrinks to a line*

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$$\psi_{j+1}(k) + \psi_{j-1}(k) = \xi_j(k)\psi_j(k)\,, \quad k \in \mathfrak{K}\,,$$

where $\psi_j(k) := \psi(j\pi, k)$ and $\xi(k)$ was introduced above, ξ_j corresponding the coupling α_j. The two equations are intimately related.

Duality, continued

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solutions to the former satisfying the δ-coupling conditions. In addition, the former belongs to $L^p(\Gamma)$ if and only if $\{\psi_j(k)\}_{j\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$, the claim being true for both $p \in \{2, \infty\}$.

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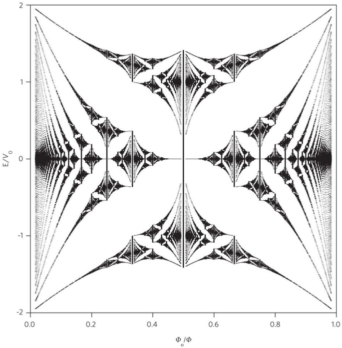

representing the spectrum of the difference operator associated with the *almost Mathieu equation*

$$u_{n+1} + u_{n-1} + 2\lambda\cos(2\pi(\omega + n\alpha))u_n = \epsilon u_n$$

for $\lambda = 1$, otherwise called *Harper equation*, as a function of α

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Only recently an experimental realization of the original concept was achieved using a graphene lattice [Dean et al'13], [Ponomarenko'13]

A chain with linear field growth

Suppose that $A_j = \alpha j + \theta$ holds for some $\alpha, \theta \in \mathbb{R}$ and every $j \in \mathbb{Z}$

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Theorem (after Pankrashkin'13)

For any interval $J \subset \mathbb{R} \setminus \sigma_D$, the operator $(H_{\gamma,A})_J$ is unitarily equivalent to the pre-image $\eta^{(-1)}\big((L_A)_{\eta(J)}\big)$, where L_A is the operator on $\ell^2(\mathbb{Z})$ acting as $(L_A q\varphi)_j = 2\cos(A_j\pi)\varphi_{j+1} + 2\cos(A_{j-1}\pi)\varphi_{j-1}$ and

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$$\eta(z) := \gamma s(\pi;z) + 2c(\pi;z) + 2s'(\pi;z)$$

Important: a simple gauge transformation shows that it is only the *fractional part of A_j* which matters. Consequently, the case of a *rational slope*, $\alpha = p/q$, is reduced to a periodic problem allowing the usual Floquet-Bloch treatment

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(b) *If $\gamma \neq 0$ and $\alpha = p/q$ with p, q relatively prime, $\alpha j + \theta + \frac{1}{2} \notin \mathbb{Z}$ for all $j = 0, \dots, q-1$, then $-\Delta_{\gamma,A}$ has infinitely degenerate ev's $\{n^2 |\, n \in \mathbb{N}\}$ interlaced with an ac part consisting of q-tuples of closed intervals*

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(c) *If the situation is as in (b) but $\alpha j + \theta + \frac{1}{2} \in \mathbb{Z}$ holds for some $j = 0, \dots, q-1$, then the spectrum $\sigma(-\Delta_{\gamma,A})$ consists of infinitely degenerate eigenvalues only, the Dirichlet ones plus q distinct others in each interval $(-\infty, 1)$ and $\big(n^2, (n+1)^2\big)$.*

The chain graph spectrum, continued

Theorem (E-Vašata'17, cont'd)

(d) *If* $\gamma \neq 0$ *and* $\alpha \notin \mathbb{Q}$, *then* $\sigma(-\Delta_{\gamma,A})$ *does not depend on* θ *and it is a disjoint union of the isolated-point family* $\{n^2 |\, n \in \mathbb{N}\}$ *and Cantor sets, one inside each interval* $(-\infty, 1)$ *and* $\big(n^2, (n+1)^2\big)$, $n \in \mathbb{N}$. *Moreover, the overall Lebesgue measure of* $\sigma(-\Delta_{\gamma,A})$ *is zero.*

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Theorem (E-Vašata'17, cont'd)

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Furthermore, using a result of [Last-Shamis'16] one can also show

Proposition

Let $A_j = \alpha j + \theta$ for some $\alpha, \theta \in \mathbb{R}$ and every $j \in \mathbb{Z}$. There exist a dense G_δ set of the slopes α for which, and all θ, the Haussdorff dimension

$$\dim_H \sigma(-\Delta_{\gamma,A}) = 0$$

Changing topic: graphs with a few gaps only

The graphs in the previous example had 'many' gaps indeed. Let us now ask whether periodic graphs can have 'just a few' gaps

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The question arises, whether it is a 'law' at all?. In other words, do infinite periodic graphs having a *finite nonzero* number of open gaps exist? From obvious reasons we would call them *Bethe–Sommerfeld graphs*

Scale-invariant coupling

The answer depends on the vertex coupling. Recall that the standard conditions for self-adjoint coupling of n edges at a vertex,

$$(U - I)\Psi + i(U + I)\Psi' = 0\,,$$

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Theorem ([E-Turek'17])

An infinite periodic quantum graph does not belong to the Bethe-Sommerfeld class if the couplings at its vertices are scale-invariant.

Proof idea

The spectrum is determined by *secular equation* [BB'15]: we define

$$F(k;\vec{\vartheta}) := \det\left(\mathbf{I} - \mathrm{e}^{i(\mathbf{A}+k\mathbf{L})}\mathbf{S}(k)\right),$$

where the $2E \times 2E$ matrices **A**, **L**, and **S** are as follows: the diagonal matrix **L** is given by the lengths of the directed edges (bonds) of Γ,

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Proof idea, and an extension

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Recall next that the vertex conditions can be equivalently written as

$$\begin{pmatrix} I^{(r)} & T \\ 0 & 0 \end{pmatrix} \Psi' = \begin{pmatrix} S & 0 \\ -T^* & I^{(n-r)} \end{pmatrix} \Psi$$

for certain r, S, and T, where $I^{(r)}$ is the identity matrix of order r; the coupling is scale-invariant if and only if the square matrix $S = 0$

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We will consider two *associated* quantum graph Hamiltonians, H with the above vertex coupling, and H_0 where we replace S by zero

A result for this associated pair

Proposition ([E-Turek'17])

For the spectra $\sigma(H)$ and $\sigma(H_0)$ the following claims hold true:

(i) *If $\sigma(H_0)$ has an open gap, then $\sigma(H)$ has infinitely many gaps.*

(ii) *If the edge lengths are rationally dependent, then the gaps of $\sigma(H)$ asymptotically coincide with those of $\sigma(H_0)$.*

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$$\mathcal{S}(k) = -I^{(n)} + 2 \begin{pmatrix} I^{(r)} \\ T^* \end{pmatrix} \left(I^{(r)} + TT^* - \frac{1}{ik} S \right)^{-1} \begin{pmatrix} I^{(r)} & T \end{pmatrix}$$

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The same is true for $\mathbf{S}(k)$, and as consequence, the spectrum at high energies is mostly determined by the scale-invariant part. $\square$

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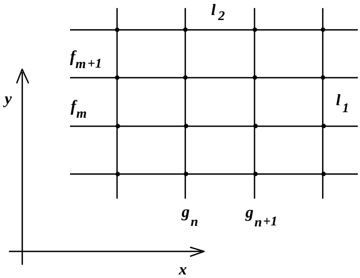
As usual with existence claims, it is enough to demonstrate an example. With this aim we are going to revisit the model of a *rectangular lattice graph* with *δ-coupling* introduced in [E'96, E-Gawlista'96]



Spectral condition

According to [E'96], a number $k^2 > 0$ belongs to a gap if and only if $k > 0$ satisfies the gap condition, which reads

$$\tan\left(\frac{ka}{2} - \frac{\pi}{2}\left\lfloor\frac{ka}{\pi}\right\rfloor\right) + \tan\left(\frac{kb}{2} - \frac{\pi}{2}\left\lfloor\frac{kb}{\pi}\right\rfloor\right) < \frac{\alpha}{2k} \quad \text{for } \alpha > 0$$

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where we denote the edge lengths ℓ_j, $j = 1, 2$, as a, b; we neglect the Kirchhoff case, $\alpha = 0$, where $\sigma(H) = [0, \infty)$.

Note that for $\alpha < 0$ the spectrum extends to the negative part of the real axis and may have a gap there, which is not important here because there is not more than a single negative gap, and this gap *always extends to positive values*

What is known

The spectrum depends on the ratio $\theta = \frac{\ell_1}{\ell_2}$. If θ is rational, $\sigma(H)$ has infinitely many gaps unless $\alpha = 0$ in which case $\sigma(H) = [0, \infty)$

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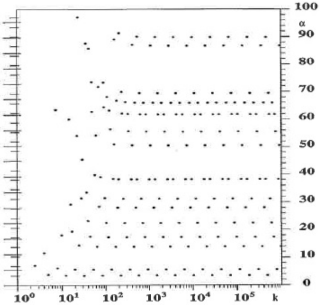

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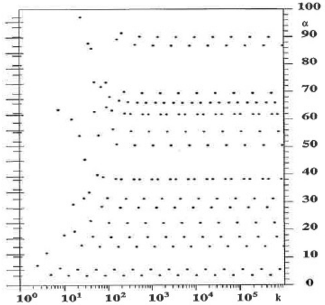

Note that they approach the limit values *from above*, also that the series open at $\frac{\pi^2}{\sqrt{5ab}}\phi^{\pm 1/2}|n^2 - m^2 - nm|,\ n, m \in \mathbb{N}$ [E-Gawlista'96]

But a closer look shows a more complex picture

Theorem ([E-Turek'17])

Let $\frac{a}{b} = \phi = \frac{\sqrt{5}+1}{2}$, *then the following claims are valid:*

(i) *If* $\alpha > \frac{\pi^2}{\sqrt{5}a}$ *or* $\alpha \leq -\frac{\pi^2}{\sqrt{5}a}$, *there are infinitely many spectral gaps.*

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Corollary

The above theorem about the existence of BS graphs is valid.

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Note that the numbers $A_j := \frac{2\pi\left(\phi^{2j}-\phi^{-2j}\right)}{\sqrt{5}}\tan\left(\frac{\pi}{2}\phi^{-2j}\right)$ form an increasing sequence the first element of which is $A_1 = 2\pi\tan\left(\frac{3-\sqrt{5}}{4}\pi\right)$ and

$$A_j < \frac{\pi^2}{\sqrt{5}} \quad \text{for all } \ j \in \mathbb{N}\,.$$

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and γ_- similarly with $\lfloor \cdot \rfloor$ replaced by $\lceil \cdot \rceil$. If the coupling constant α satisfies

$$\gamma_\pm < \pm\alpha < \frac{\pi^2}{\max\{a, b\}} \mu(\theta)\,,$$

then there is a nonzero and finite number of gaps in the positive spectrum.

Another application of periodic quantum graphs

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This motivated us to investigate situations where such an *inherent rotation* can take place, not at the graphs edges but in its *vertices*

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Componentwise, with $\psi_j = \psi_j(0+)$ and $\psi'_j = \psi'_j(0+)$, we have

$$(\psi_{j+1} - \psi_j) + i(\psi'_{j+1} + \psi'_j) = 0\,, \quad j \in \mathbb{Z} \pmod N\,,$$

which is non-trivial only for $N \geq 3$ and obviously *non-invariant w.r.t. the reverse in the edge numbering order*

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It might seem that the transport becomes trivial at small and high energies, since $\lim_{k\to 0} S(k) = -I$ and $\lim_{k\to\infty} S(k) = I$

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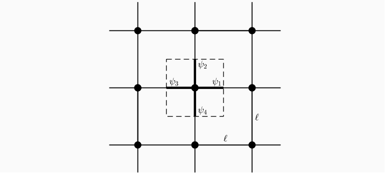

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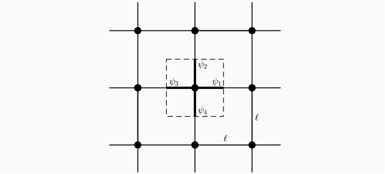

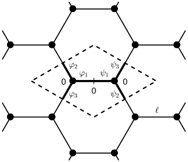

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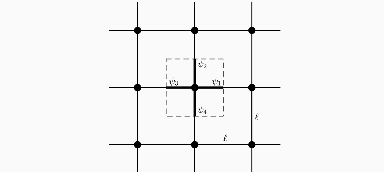

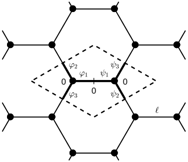

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$$16i\,\mathrm{e}^{i(\theta_1+\theta_2)}\,k\,\sin k\ell\big[(k^2-1)(\cos\theta_1+\cos\theta_2)+2(k^2+1)\cos k\ell\big]=0$$

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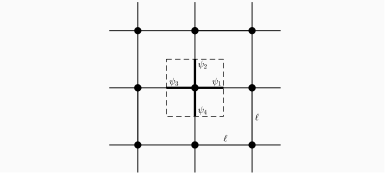

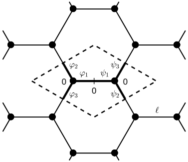

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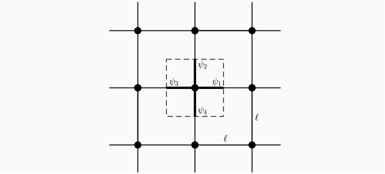
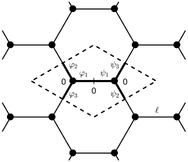
Comparison of two lattices





Spectral condition for the cases are easy to derive,

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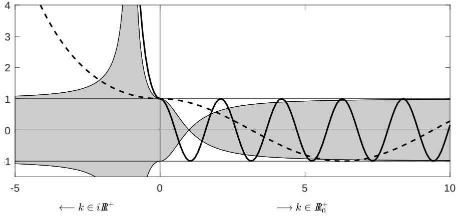
However, we can present the band solution in a *graphical form*

Picture worth of thousand words

For the two lattices, respectively, we get (with $\ell = \frac{3}{2}$, dashed $\ell = \frac{1}{4}$)



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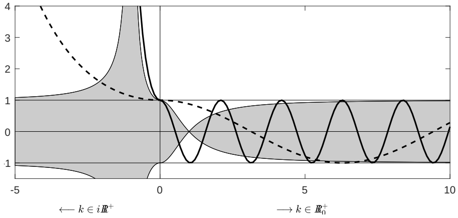

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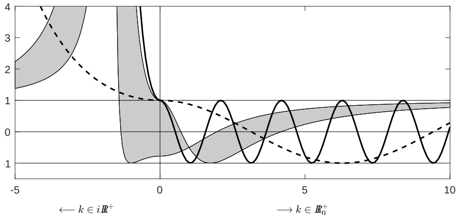

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More generally on vertex coupling symmetries

A *symmetry* is described by an invertible map in the space of boundary values, $\Theta : \mathbb{C}^n \to \mathbb{C}^n$. A vertex coupling is *symmetric w.r.t* Θ if the usual matching condition is equivalent to

$$(U - I)\Theta\Psi(0) + i(U + I)\Theta\Psi'(0) = 0\,,$$

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- *Time-reversal-invariant couplings:* in this case Θ is replaced by the antilinear operator of complex conjugation, and using relations $U^t\bar{U} = \bar{U}U^t = I$ we find easily that the matrix describing the coupling must be now invariant w.r.t. transposition, $U = U^t$

Symmetries, continued

Our focus here is on *rotationally symmetric* vertex coupling, i.e. those independent of cyclic permutations of the entries of $\Psi(0)$ and $\Psi'(0)$. They correspond to

$$\Theta = R := \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

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Note also that while the notions of permutation-invariant and time-reversal-invariant couplings are universal in the sense that they do *not* require embedding in an ambient space, the other two mentioned above make sense only if we think of the graph Γ as of *an embedded object*

Circulant matrices

Matrices satisfying $\Theta^{-1}U\Theta = U$ for the above $\Theta = R$ are *circulant matrices*, which generally take the form

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & c_{n-2} \\ \vdots & c_{n-1} & c_0 & \ddots & \vdots \\ c_2 & & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix},$$

being a particular case of Toeplitz matrices. The first row, i.e. the vector $c = (c_0, c_1, \ldots, c_{n-1})$, is called *generator* of C

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Let us recall basic properties of such matrices. The vectors

$$v_k = \frac{1}{\sqrt{n}} \left(1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k}\right)^T, \quad k = 0, 1, \ldots, n-1,$$

with $\omega := \mathrm{e}^{2\pi i/n}$ are its normalized eigenvectors corresponding to

$$\lambda_k = c_0 + c_1\omega^k + c_2\omega^{2k} + \cdots + c_{n-1}\omega^{(n-1)k}, \quad k = 0, 1, \ldots, n-1.$$

Circulant matrices, continued

Every circulant matrix C is diagonalized by the *Discrete Fourier Transform* matrix

$$F = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \dots & \omega^{(n-1)^2} \end{pmatrix},$$

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$$c_j = \frac{1}{n}\left(\lambda_0 + \lambda_1\omega^{-j} + \lambda_2\omega^{-2j} + \dots + \lambda_{n-1}\omega^{-(n-1)j}\right)$$

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A circulant matrix is unitary if and only if $|\lambda_j| = 1$ holds for all indices $j = 0, 1, \dots, n-1$. Consequently, circulant unitary matrices of order n are parametrized by an *n-tuple of real numbers*

A class of interpolating couplings

Our aim is now to investigate a class of couplings interpolating between the 'maximally rotational' one discussed above and the usual δ coupling referring to

$$U = -I + \frac{2}{n + i\alpha} J \,,$$

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We will construct them using their eigenvalues. For the circulant matrix $U(1)$ we get

$$\lambda_k = \omega^k \quad \text{if } \ k = 0, 1, \ldots, n-1$$

A class of interpolating couplings

Similarly, the eigenvalues of $U(0)$ are

$$\lambda_k = -1 + \frac{2}{n+i\alpha}\sum_{j=0}^{n-1}\omega^{jk} = \begin{cases} \frac{n-i\alpha}{n+i\alpha} & \text{for } k=0\,; \\ -1 & \text{for } k \geq 1\,. \end{cases}$$

For brevity, let us set $\frac{n-i\alpha}{n+i\alpha} = \mathrm{e}^{-i\gamma}$; note that $\gamma \begin{cases} = 0 & \text{for } \alpha = 0; \\ \in (0,\pi) & \text{for } \alpha > 0; \\ \in (-\pi, 0) & \text{for } \alpha < 0. \end{cases}$

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$$c_j(t) = \frac{1}{n}\left(\mathrm{e}^{-i(1-t)\gamma} - \sum_{k=1}^{n-1}\mathrm{e}^{i\pi t\left(\frac{2k}{n}-1\right)}\cdot\omega^{-kj}\right)$$

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$$\det[(U(t) - I) - i\kappa(U(t) + I)] = 0\,,$$

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$$\kappa = -\tan\frac{(1-t)\gamma}{2} \quad \text{for } \alpha < 0 \text{ and } t \neq 1$$

$$\kappa = -\cot\left(\frac{j}{n} - \frac{1}{2}\right)\pi t \quad \text{for } n \geq 3 \text{ and } j < \tfrac{n}{2}$$

These solutions give rise to the eigenvalues $-\kappa^2 < 0$ of the star graph:

- there is a negative eigenvalue $-\tan^2\frac{(1-t)\gamma}{2}$ whenever $\alpha < 0$;
- if $n \geq 3$, there is an additional $\lfloor\frac{n-1}{2}\rfloor$-tuple for aech $t \in (0,1]$, namely $-\cot^2\left(\frac{j}{n}-\frac{1}{2}\right)\pi t$ with j running through $1,\dots,\frac{n-1}{2}$ for n odd and $1,\dots,\frac{n}{2}-1$ for n even

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- If $t \to 1-$, the eigenvalues $-\kappa^2$ approach zero and $-\tan^2 \frac{j}{n}$, respectively. When $t = 1$, the only negative eigenvalues are $-\tan^2 \frac{j}{n}$ with j taking values $1, \dots, \frac{n-1}{2}$ for n odd and $1, \dots, \frac{n}{2} - 1$ for n even – as we have seen before

Scattering

Knowing $U(t)$ we easily find the on-shell S-matrix. In particular, its eigenvalues are

$$\mu_j(t) = \frac{k-1+(k+1)\lambda_j(t)}{k+1+(k-1)\lambda_j(t)} \quad \text{for } j = 0, 1, \ldots, n-1.$$

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- If n is odd, then $\lim_{k\to\infty} \mu_j = 1$ for all $j = 0, 1, \dots, n-1$, hence

$$\lim_{k\to\infty} \mathcal{S}(k) = I$$

- If n is even, then $\lim_{k\to\infty} \mu_j = 1$ for $j \neq \frac{n}{2}$, while $\lim_{k\to\infty} \mu_{n/2} = -1$, and consequently, the generator of $\lim_{k\to\infty} \mathcal{S}(k)$ equals

$$\left(1 - \frac{2}{n}, \frac{2}{n}, -\frac{2}{n}, \frac{2}{n}, \dots, -\frac{2}{n}, \frac{2}{n}\right)$$

As in the 'extreme' rotational case, the S-matrix behaves at high energies differently for odd and even n

Square lattice

We consider again a lattice of spacing ℓ, now with the interpolating coupling. The treatment is similar to the particular case analyzed earlier; it yields the spectral condition

$$512\,\mathrm{e}^{i(\theta_1+\theta_2)}\,\mathrm{e}^{-i\frac{(1-t)\gamma}{2}}\left[V_3k^3+V_2k^2+V_1k+V_0\right]=0\,,$$

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where

$$V_3=-\cos\frac{(1-t)\gamma}{2}\sin^2\frac{\pi t}{4}\sin k\ell(\cos\theta_1+\cos\theta_2+2\cos k\ell)\,;$$

$$V_2=2\sin\frac{(1-t)\gamma}{2}\sin^2\frac{\pi t}{4}(\cos\theta_1+\cos k\ell)(\cos\theta_2+\cos k\ell)\,;$$

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Hence k^2 belongs to the spectrum *iff* there are $\theta_1,\theta_2\in[-\pi,\pi)$ such that

$$V_3k^3+V_2k^2+V_1k+V_0=0\,.$$

The case $\alpha = 0$

If $\alpha = 0$ we have $V_2 = V_0 = 0$ and the spectral condition reduces to

$$\sin k\ell \left[\left(k^2 \sin^2 \frac{\pi t}{4} - \cos^2 \frac{\pi t}{4} \right) (\cos\theta_1 + \cos\theta_2) - 2 \left(k^2 \sin^2 \frac{\pi t}{4} + \cos^2 \frac{\pi t}{4} \right) \cos k\ell \right] = 0 \,.$$

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The condition describing spectral bands can rewritten as

$$\left| k^2 \sin^2 \frac{\pi t}{4} - \cos^2 \frac{\pi t}{4} \right| \geq \left(k^2 \sin^2 \frac{\pi t}{4} + \cos^2 \frac{\pi t}{4} \right) |\cos k\ell| \,.$$

Note that if $t = 0$ (i.e., pure Kirchhoff), the condition simplifies to $|\cos k\ell| \leq 1$, which is satisfied for all $k \geq 0$. Putting this trivial case aside, from now on we assume that $t \in (0, 1]$.

The case $\alpha = 0$, continued

For $t \in (0,1]$ the spectrum obviously contains infinitely many gaps located in the vicinity of $k = \frac{m\pi}{\ell}$ for $m \in \mathbb{N}$. The band condition for $k > 0$ can be rewritten as

$$\left(k\left|\tan\frac{k\ell}{2}\right| - \cot\frac{\pi t}{4}\right)\left(k\left|\cot\frac{k\ell}{2}\right| - \cot\frac{\pi t}{4}\right) \geq 0$$

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and the zeros of the factors correspond to the band edges. The latter intersect at points where $k\left|\tan\frac{k\ell}{2}\right| = k\left|\cot\frac{k\ell}{2}\right| = \cot\frac{\pi t}{4}$, i.e., at (t,k) such that

$$k = \left(m - \frac{1}{2}\right)\frac{\pi}{\ell} \quad \text{and} \quad t = \frac{4}{\pi}\mathrm{arccot}\left(m - \frac{1}{2}\right)\frac{\pi}{\ell}, \quad m \in \mathbb{N}.$$

At those values of t the spectrum shrinks into a 'flat band'

The case $\alpha = 0$, continued

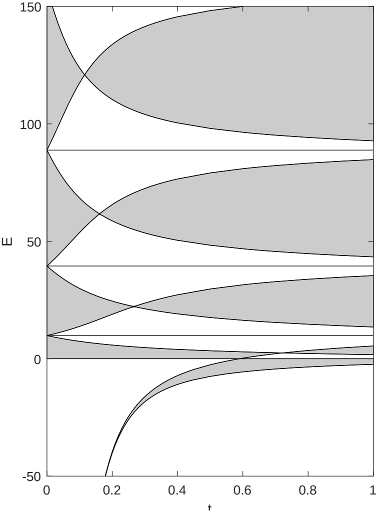

Figure: Interpolation with the Kirchhoff coupling for $\ell = 1$, the spectrum is indicated by the shaded regions.

The case $\alpha = 0$, continued

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so it can be satisfied only for k located in small neighborhoods of $\frac{m\pi}{\ell}$. The approximate width of the m-th spectral gap is equal to

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in the momentum variable, hence in energy the gaps are asymptotically constant, of the widths $\frac{8}{\ell}\cot\frac{\pi t}{4} + \mathcal{O}(m^{-1})$. Note that the gap width increases as t diminishes from this extreme value but eventually it starts to decrease again, to the point of vanishing at $t = 0$.

The general case, $\alpha \in \mathbb{R}$

The analysis follows the same line but becomes more complicated; the spectral condition now yields spectra, as functions of t, in the form of a *union of pairs of regions*

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Let us summarize our observations about the band spectra of our lattices:

(i) *A 'discontinuity' at $t = 0$:* There is always a spectral band which becomes narrow and strongly negative as $t \to 0$ and eventually disappears. This obviously corresponds to the behavior of the 'additional' eigenvalues of a single vertex

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(ii) *Point degeneracies for $\alpha = 0$:* In the Kirchhoff case spectral bands may collapse to a point at particular values of t. It is not the case if $\alpha \neq 0$ where the spectral condition has other solutions which smear these Kirchhoff degenerate eigenvalues into bands of nonzero width, more pronounced as we are going farther from the Kirchhoff case

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(iii) *Non-monotonicity of the gap widths:* As mentioned already the widths of the gaps are not monotonous with respect to the interpolation parameter, the same is true if $\alpha \neq 0$

Illustration, weakly attractive interaction

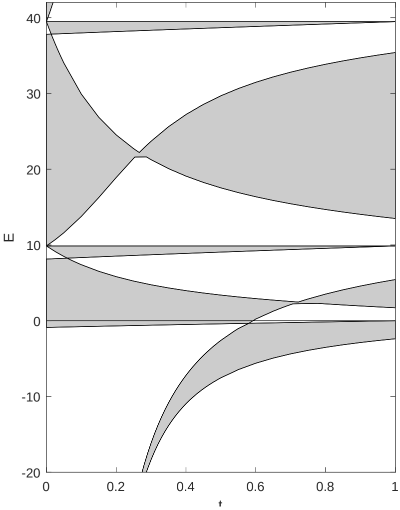

Figure: Interpolation with an attractive δ coupling, $\alpha = -4(\sqrt{2}-1)$, $\ell = 1$.

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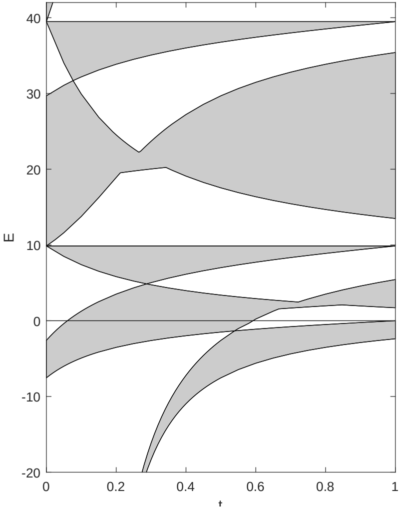

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Other properties

(iv) *α-independence of some bands:* Some curves marking the band edges are independent of α. However, for $\alpha \neq 0$ the 'band edges' coincide only in parts of the interval $[0, 1]$, as there is a neighborhood of $t = \frac{4}{\pi}\mathrm{arccot}\,\frac{3\pi}{2}$ in which there is an additional spectrum

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(vi) *Flat band spreading:* Infinitely degenerate eigenvalues for $t = 1$ may smear when the interpolation parameter decreases, but we know that they also remain in the spectrum. A possible explanation may come from the presence of different 'elementary' eigenfunctions in the case $t = 1$, namely 'Dirichlet' and 'Neumann' type, which could behave differently with respect to t

Illustration, larger lattice spacing

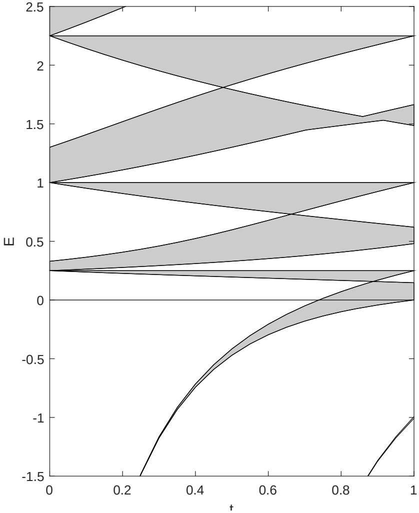

Figure: Interpolation with an attractive δ coupling, $\alpha = -4(\sqrt{2} + 1)$ and $l = 2\pi$.

And finally, band edges positions

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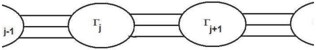

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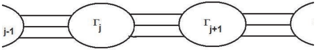

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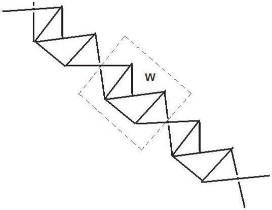

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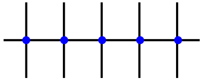

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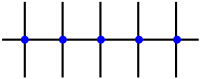

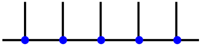

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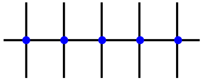

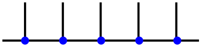

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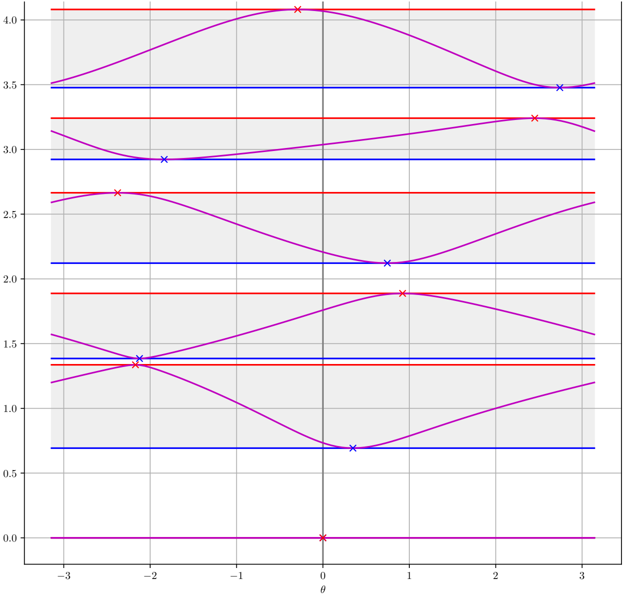
Two-sided comb: dispersion curves



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Thank you for your attention!