Constrained evolution problems on a metric graph

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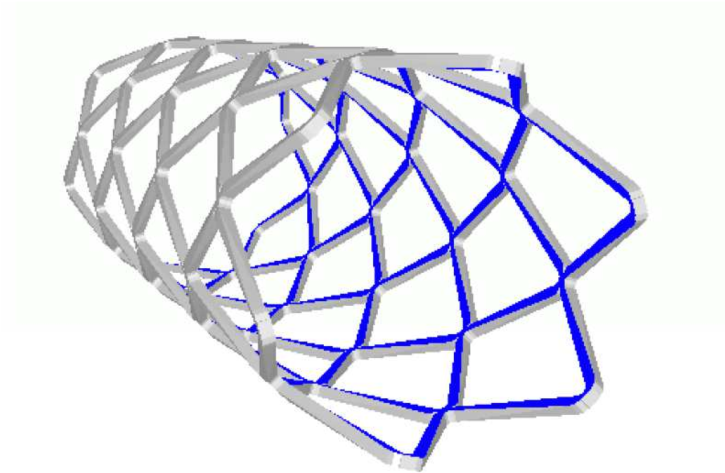
Differential Operators on Graphs and Waveguides, TU Graz, 2019.

[1] joint work with M. Ljulj, V. Mehrmann and J. Tambača.

This is the story of two discretizations



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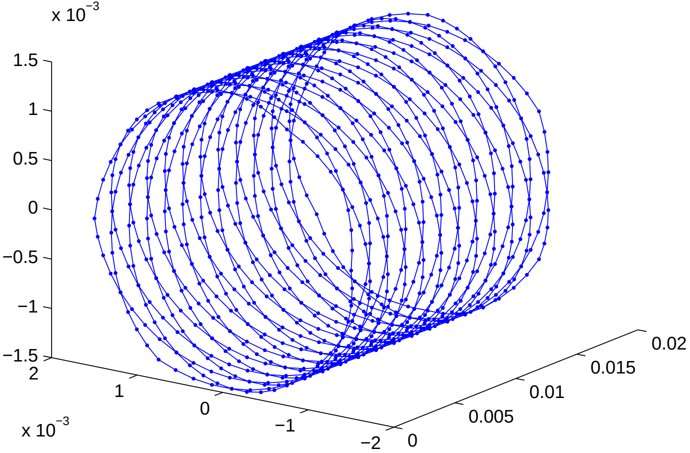

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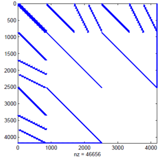
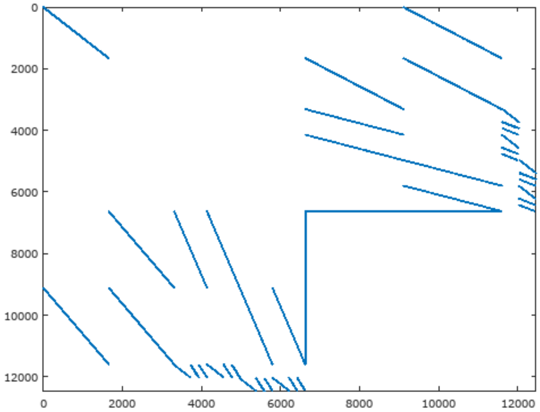




This is the story of two discretizations

- Which would you rather have?

A more comprehensive list of collaborators

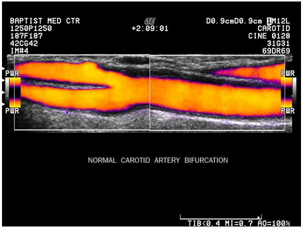
- Josip Tambača, Department of Mathematics, University of Zagreb (thanks for the slides).
- Sunčica Čanić, UC Berkeley
- David Paniagua, Baylor College of Medicine, Huston
- Bojan Žugec, Faculty of Organization and Informatics, University of Zagreb
- Mate Kosor, Maritime Department, University of Zadar
- Matko Ljulj, University of Zagreb
- Josip Iveković, University of Zagreb
- Matea Galović, University of Zagreb
- Marko Hajba, University of Zagreb
- K. Schmidt, TU Darmstadt
- Volker Mehrmann, TU Berlin

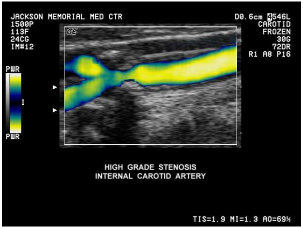
Outline

1. About stents
   - Usage of stents
   - Stent properties
2. FEM on the metric graph
   - 1D curved rod model
   - 1D stent model
   - Weak formulation
   - Mixed formulation
3. Time dependent problems
   - Mixed formulation
   - Comparison of the 1D and 3D model
   - Examples
4. Some further developments
   - Optimal design

Usage of stents

- carotid stenosis





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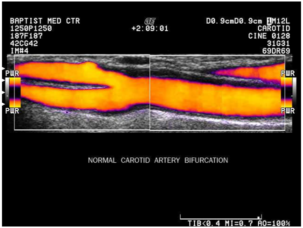
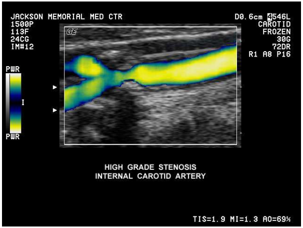
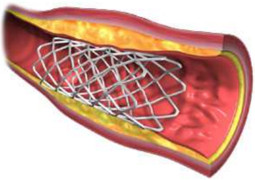
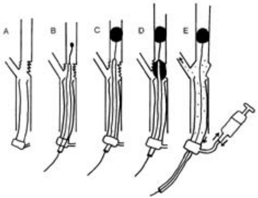
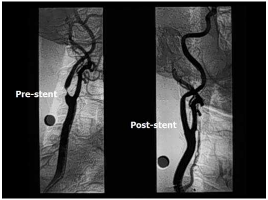


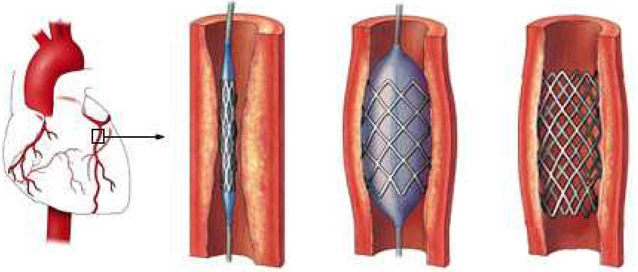


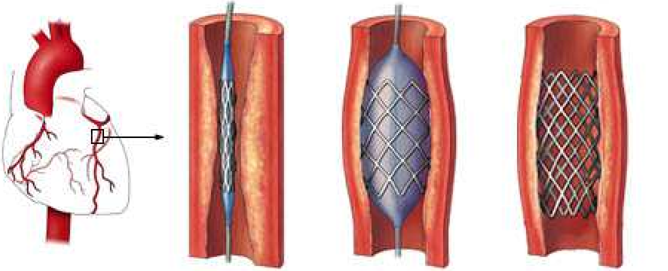
- stent: "solution" of the problem

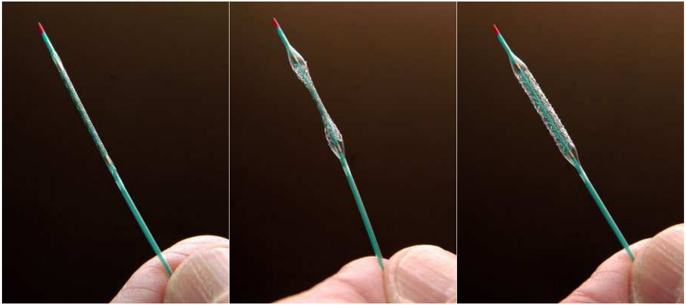












Stent properties

- made of cylindrical tubes by laser cuts
- mostly made of metals: 316L stainless steel, lately from cobalt, chrome and nickel.
- expanded on the place of stenosis (balloon expandable is dominant (99%) over self-expanding)
- properties depend on
  - complex geometry of stent,
  - mechanical properties of material.
- metal $\Longrightarrow$ theory of elasticity

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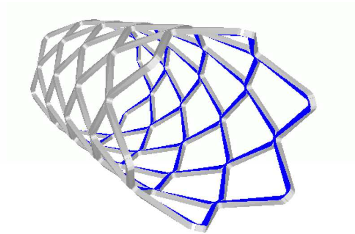
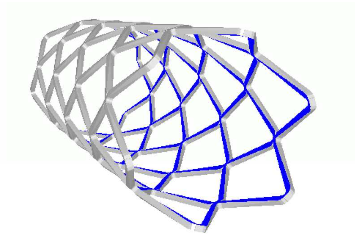
- metal $\Longrightarrow$ theory of elasticity
- small deformation $\Longrightarrow$ use linearized elasticity
- We are looking for stents such that response of the stented artery is closest to the response of the healthy artery.



- stent is a 3D elastic body
- struts thin $\Longrightarrow$ very fine mesh needed



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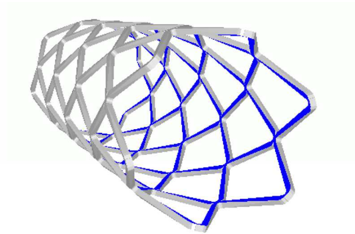

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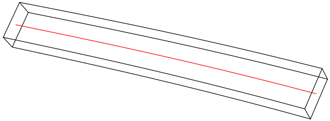

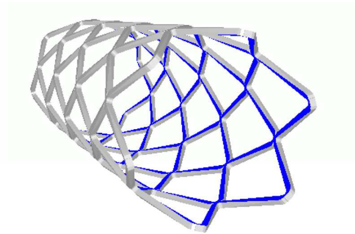

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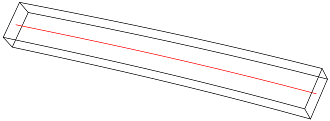
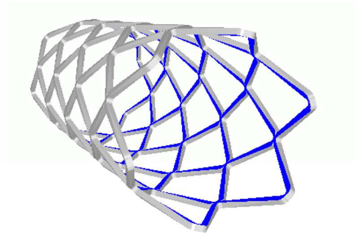


- use simpler model: 1D curved rod model

(rigorous justification Jurak, Tambača (1999), (2001))



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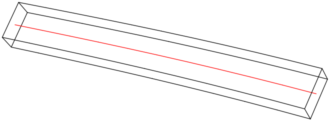

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1D curved rod model

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- $\boldsymbol{t}$ – tangent, $\boldsymbol{n}$ – normal, $\boldsymbol{b}$ – binormal for middle curve
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- 1D model much easier and quicker to solve than 3D
  - numerical approximation in 3D: minutes
  - numerical approximation in 1D: seconds

1D stent model

- At each edge: the system of 12 ODE

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  - for dynamical quantities: $\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{q}}$
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rigorous mathematical justification by Γ-convergence in nonlinear elasticity (Tambača, Velčić (2010), Griso (2010))

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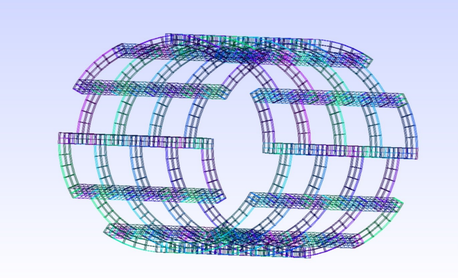
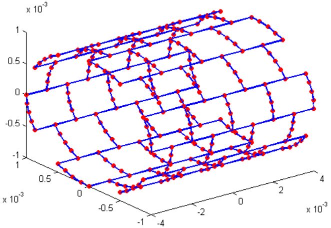
(Tambača, Kosor, Čanić, Paniagua, SIAM J. Appl. Math., 2010)

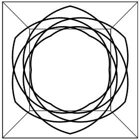
(Zunino, Tambača, Cutri, Čanić, Formaggia, Migliavacca, Annals of Biomedical Engineering, 2015)

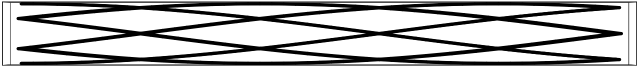
Outline in two pictures





1D stent model

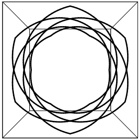


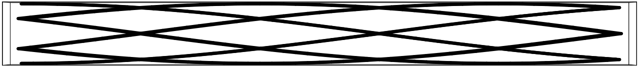


For the stent model we need:

- $\mathcal{V}$ – vertices ($n_{\mathcal{V}}$ number of vertices)

1D stent model





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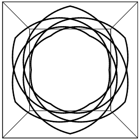

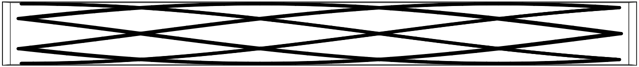

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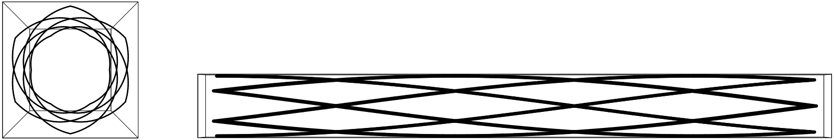
1D stent model



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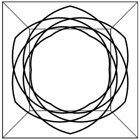

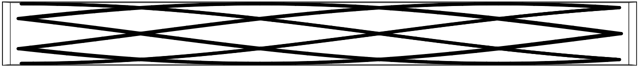

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- parametrization of edges $\mathbf{\Phi}_e : [0, \ell_e] \to \mathbb{R}^3$ (for $\boldsymbol{t}^e$ and $\mathbf{Q}^e$)
- material and cross–section properties (for $\mathbf{H}^e$)

Two options

1. Place the contact conditions in the Sobolev space on a graph
2. Consider a larger product Sobolev space and leave contact conditions in vertices as constraints

1D stent model – in $H^1(\mathcal{N}; \mathbb{R}^6)$

Unknown: $\boldsymbol{U} = (\boldsymbol{U}^1, \ldots, \boldsymbol{U}^{n_\mathcal{E}}) = ((\tilde{\boldsymbol{u}}^1, \tilde{\boldsymbol{\omega}}^1), \ldots, (\tilde{\boldsymbol{u}}^{n_\mathcal{E}}, \tilde{\boldsymbol{\omega}}^{n_\mathcal{E}}))$
Test function: $\boldsymbol{V} = (\boldsymbol{V}^1, \ldots, \boldsymbol{V}^{n_\mathcal{E}}) = ((\tilde{\boldsymbol{v}}^1, \tilde{\boldsymbol{w}}^1), \ldots, (\tilde{\boldsymbol{v}}^{n_\mathcal{E}}, \tilde{\boldsymbol{w}}^{n_\mathcal{E}}))$

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Function spaces on the graph $\mathcal{N}$:

$$\begin{aligned} H^1(\mathcal{N};\mathbb{R}^6) =& \Big\{ \boldsymbol{V} \in \prod_{i=1}^{n_\mathcal{E}} H^1((0,\ell^{e_i});\mathbb{R}^6) : \\ & \qquad \boldsymbol{V}^i((\boldsymbol{\Phi}^i)^{-1}(v)) = \boldsymbol{V}^j((\boldsymbol{\Phi}^j)^{-1}(v)), v \in \mathcal{V}, v \in e^i \cap e^j \Big\}, \\ V_{\text{stent}} =& \{ \boldsymbol{V} \in H^1(\mathcal{N};\mathbb{R}^6) : \tilde{\boldsymbol{v}}^{i\prime} + \boldsymbol{t}^i \times \tilde{\boldsymbol{w}}^i = 0, i = 1, \ldots, n_\mathcal{E}, \int_\mathcal{N} \boldsymbol{V} = 0 \}. \end{aligned}$$

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Function spaces on the graph $\mathcal{N}$:

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H^1(\mathcal{N};\mathbb{R}^6) = & \Big\{ \boldsymbol{V} \in \prod_{i=1}^{n_\mathcal{E}} H^1((0,\ell^{e_i});\mathbb{R}^6) : \\
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Add weak formulations for rods:

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find $\boldsymbol{U} \in V_{\mathrm{stent}}$ such that

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(Čanić & Tambača, IMA Journal of Applied Mathematics, 2012)

1D stent model: properties for the forms

$$a(\boldsymbol{U}, \boldsymbol{V}) = \sum_{i=1}^{n_{\mathcal{E}}} \int_0^{\ell^i} \mathbf{Q}^i \mathbf{H}^i (\mathbf{Q}^i)^T (\tilde{\boldsymbol{\omega}}^i)' \cdot \tilde{\boldsymbol{w}}' dx_1, \quad f(\boldsymbol{V}) = \sum_{i=1}^{n_{\mathcal{E}}} \int_0^{\ell^i} \tilde{\boldsymbol{f}}^i \cdot \tilde{\boldsymbol{v}}^i dx_1,$$

Problem (W)

Find $\boldsymbol{U} \in V_{\text{stent}}$ *such that*

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Lax-Milgram implies

Theorem (form a is V_{stent}-elliptic)

There exits a unique solution of Problem (W).

1D stent model: a look into constraints

Problem: to construct functions within V_{stent}!

$$V_{\mathrm{stent}} = \{\boldsymbol{V} \in H^1(\mathcal{N};\mathbb{R}^6) : \tilde{\boldsymbol{v}}^{i\prime} + \boldsymbol{t}^i \times \tilde{\boldsymbol{w}}^i = 0, i = 1, \ldots, n_{\mathcal{E}}, \int_{\mathcal{N}} \boldsymbol{V} = 0\},$$

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$$+ \boldsymbol{\alpha} \cdot \sum_{i=1}^{n_{\mathcal{E}}} \int_0^{\ell^i} \tilde{\boldsymbol{v}}^i dx_1 + \boldsymbol{\beta} \cdot \sum_{i=1}^{n_{\mathcal{E}}} \int_0^{\ell^i} \tilde{\boldsymbol{w}}^i dx_1,$$

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Then

$$V_{\mathrm{stent}} = \{\boldsymbol{V} \in H^1(\mathcal{N};\mathbb{R}^6) : b(\boldsymbol{V},\boldsymbol{\Theta}) = 0, \forall \boldsymbol{\Theta} \in M\}.$$

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Solution: mixed formulation

Mixed formulation

Problem (M)

Find $(\boldsymbol{U}, \boldsymbol{P}) \in H^1(\mathcal{N}; \mathbb{R}^6) \times M$ *such that*

$$a(\boldsymbol{U}, \boldsymbol{V}) + b(\boldsymbol{V}, \boldsymbol{P}) = f(\boldsymbol{V}), \qquad \boldsymbol{V} \in H^1(\mathcal{N}; \mathbb{R}^6),$$
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\end{aligned}
$$

Theorem

If b satisfies the inf-sup condition:

$$
\inf_{\boldsymbol{\Theta} \in L^2(\mathcal{N};\mathbb{R}^3)\times\mathbb{R}^3\times\mathbb{R}^3} \; \sup_{\boldsymbol{V} \in H^1(\mathcal{N};\mathbb{R}^6)} \frac{b(\boldsymbol{V}, \boldsymbol{\Theta})}{\|\boldsymbol{V}\|_{H^1(\mathcal{N};\mathbb{R}^6)} \|\boldsymbol{\Theta}\|_{L^2(\mathcal{N};\mathbb{R}^3)\times\mathbb{R}^3\times\mathbb{R}^3}} \geq \beta > 0
$$

the Problem (M) *has unique solution.*
Then the Problem (W) *is equivalent to Problem* (M)*!*

Alternatively – in the direct product space

We will also consider direct product space formulation

$\Rightarrow$ Let linear algebra solver do the heavy lifting

- We build $H^1(\mathcal{N};\mathbb{R}^6)$ by eliminating constraints

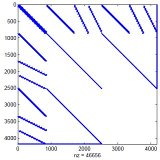
$$\boldsymbol{V}^i((\boldsymbol{\Phi}^i)^{-1}(v)) = \boldsymbol{V}^j((\boldsymbol{\Phi}^j)^{-1}(v))$$

- Alternative introduce new variables and extend the restriction form. We then we get

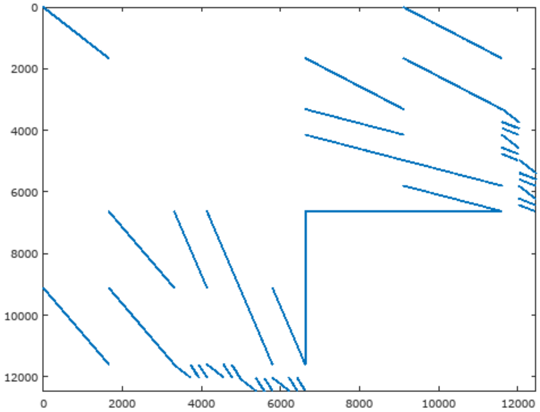
$$V = L^2(\mathcal{N};\mathbb{R}^3) \times L^2(\mathcal{N};\mathbb{R}^3) \times \mathbb{R}^{3n_\mathcal{E}} \times \mathbb{R}^{3n_\mathcal{E}} \times \mathbb{R}^{3n_\mathcal{E}} \times \mathbb{R}^{3n_\mathcal{E}} \times \mathbb{R}^3 \times \mathbb{R}^3,$$
$$M = L^2_{H^1}(\mathcal{N};\mathbb{R}^3) \times L^2_{H^1}(\mathcal{N};\mathbb{R}^3) \times \mathbb{R}^{3n_\mathcal{V}} \times \mathbb{R}^{3n_\mathcal{V}}.$$

- Projection by interpolation theory will be easier, since this is just a large product space
- We pay by considerably increasing the dimension of the problem – is it to expensive?

Recall



0
500
1000
1500
2000
2500
3000
3500
4000
0
1000
2000
3000
4000
nz = 46656



0
2000
4000
6000
8000
10000
12000
0
2000
4000
6000
8000
10000
12000

inf–sup for stents

Lemma

All stents in class $\mathcal{S}$ satisfy inf–sup *condition.*

Class $\mathcal{S}$ contains:

- stents with all curved struts
- stents with straight struts which are linearly independent in all vertices they meet.
- Proof, essentially LA (Grubišić, Iveković, Tambača, Žugec, Rad HAZU, 2017)
- Proof for the V space formulation (Grubišić, Ljulj, Mehrmann, Tambača, 2018)
- Direct product space formulation adds additional constraints (continuity at vertices of displacements and couples)

FEM for mixed formulation

Let us take finite dimensional subspaces

$$V^h \subset H^1(\mathcal{N};\mathbb{R}^6), \qquad M^h \subset M.$$

Problem (M^h)

Find $(\boldsymbol{U}^h, \boldsymbol{P}^h) \in V^h \times M^h$ *such that*

$$\begin{aligned} a(\boldsymbol{U}^h, \boldsymbol{V}^h) + b(\boldsymbol{V}^h, \boldsymbol{P}^h) &= f(\boldsymbol{V}^h), \qquad \boldsymbol{V}^h \in V^h, \\ b(\boldsymbol{U}^h, \boldsymbol{\Theta}^h) = 0, &\qquad \boldsymbol{\Theta}^h \in M^h. \end{aligned}$$

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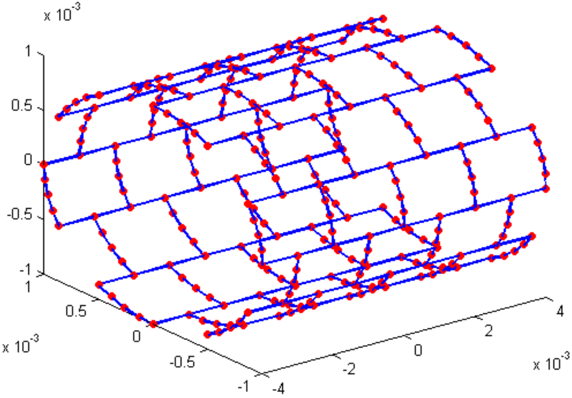
If $(\boldsymbol{U}^h, \boldsymbol{P}^h)$ solves Problem (M^h) then $\boldsymbol{U}^h$ solves weak forulation

Problem (W^h)

Find $\boldsymbol{U}^h \in V^h_{\mathrm{stent}} = \{\boldsymbol{V}^h \in V^h : b(\boldsymbol{V}^h, \boldsymbol{\Theta}^h) = 0, \boldsymbol{\Theta}^h \in M^h\}$ *such that*

$$a(\boldsymbol{U}^h, \boldsymbol{V}^h) = f(\boldsymbol{V}^h), \qquad \boldsymbol{V}^h \in V^h_{\mathrm{stent}}.$$

Geometry matters!



Existence for $\boldsymbol{U}^h$

Note: $V^h_{\mathrm{stent}} \not\subset V_{\mathrm{stent}}$

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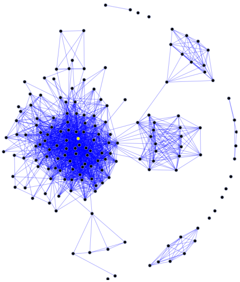
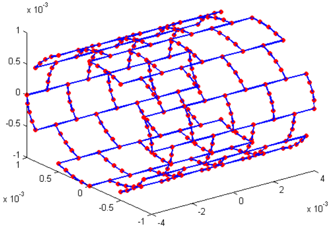
Remark

We miss discrete inf-sup for the formulation in $H^1(\mathcal{N}; \mathbb{R}^6)$. However, for the direct product space formulation inf-sup follows readily!

Note – compare with information science PDE graph models.





- Information science:
  - Huge graph, but a scalar function
  - “Finite difference” discretization
- Structural optimization:
  - Highly structured graph, nodes of low incidence degree
  - Constrained vector valued functions.
  - FEM discretization

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and discrete inextensibility

$$\int_0^{\ell^i} \tilde{\boldsymbol{\theta}}^i \cdot ((\tilde{\boldsymbol{u}}^i)' + \boldsymbol{t}^i \times \tilde{\boldsymbol{\omega}}^i) dx_1 = 0, \qquad \tilde{\boldsymbol{\theta}}^i \in P^1.$$

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Note that full inextensibility poses to big restriction for the error estimate!

Convergence of FEM for stents

- The interpolation satisfies the estimate

$$\|\boldsymbol{U} - \mathcal{I}^2 \boldsymbol{U}\|_{H^1(0,\ell^i;\mathbb{R}^6)} \leq Ch^2 \|\boldsymbol{U}'''\|_{L^2(0,\ell^i)}.$$

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Adapting an argument from Boffi, Brezzi and Gastaldi we obtain

Theorem

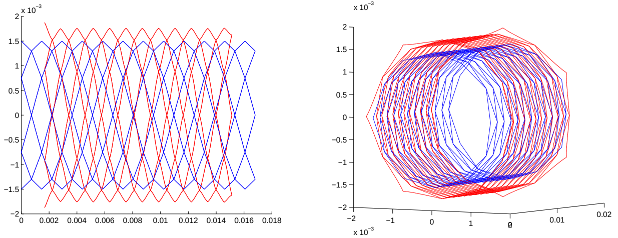
Let $h > 0$ denotes the size of the discretization mesh, $(\boldsymbol{U}, \boldsymbol{P})$ is the solution of the mixed formulation and $(\boldsymbol{U}^h, \boldsymbol{P}^h)$ solution of the discretized problem piecewisely $(P^2)^6 \times (P^1)^3$ polynomials. Then

$$\|\boldsymbol{U} - \boldsymbol{U}^h\|_{H^1(\mathcal{N};\mathbb{R}^6)} \leq Ch^2 (\|\boldsymbol{U}'''\|_{L^2(\mathcal{N};\mathbb{R}^6)} + \|\boldsymbol{P}''\|_{L^2(\mathcal{N};\mathbb{R}^3)}).$$

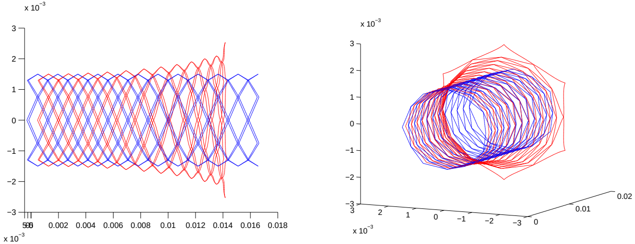
- no error estimate for multipliers!
- yields resolvent estimates for the spectral problem!

(Grubišić, Tambača, in review NLAA)

Numerical examples – convergence validate estimates

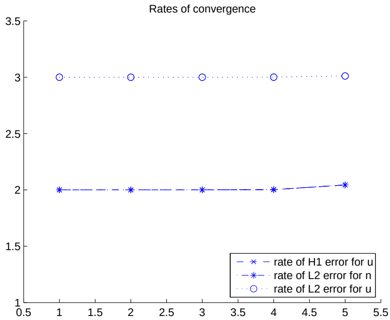


constant radial forcing



x_1^2 radial forcing

Numerical rate of convergence



- $E(h) = Ch^{\alpha}$
- for radial forcing depending as x_1^2 on the longitudinal variable.

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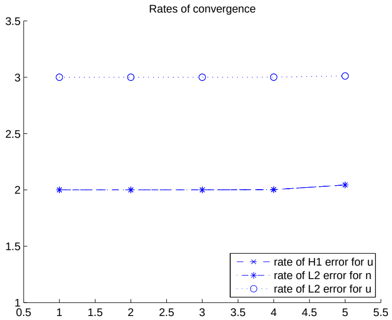

- $E(h) = Ch^{\alpha}$
- for radial forcing depending as x_1^2 on the longitudinal variable.
- numerical approximations for 2, 4, 8, 16, 32, 64 division of struts are compared with solution for 128 divisions of every strut.
- L^2 errors are one order better then H^1.
- coincide with theoretical estimates.

A paradox!

	direct product for.		in $H^1(\mathcal{N})$	
splitting #	time(s)	size of matrix	time(s)	size of matrix
8	22	105198	2	38958
16	47	211182	42	78702
32	108	423150	152	158190
64	288	847086	629	317166
128	903	1694958	4183	635118

Tablica: Times and matrix sizes for the old and new numerical scheme

Evolution problem

$$
\begin{aligned}
\tilde{\boldsymbol{p}}^{i\prime} + \tilde{\boldsymbol{f}}^{i} &= \rho^i A^i \partial_{tt} \tilde{\boldsymbol{u}}^i, \\
\tilde{\boldsymbol{q}}^{i\prime} + \boldsymbol{t}^i \times \tilde{\boldsymbol{p}}^i &= 0, \\
\tilde{\omega}^{i\prime} + \mathbf{Q}^i \mathbf{H}^i (\mathbf{Q}^i)^T \tilde{\boldsymbol{q}}^i &= 0, \\
\tilde{\boldsymbol{u}}^{i\prime} + \boldsymbol{t}^i \times \tilde{\omega}^i &= \boldsymbol{\theta}^i,
\end{aligned}
\qquad i = 1, \ldots, n_{\mathcal{E}} \qquad + \text{ junction conditions}
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Let (limit of 3D linearized Antman-Cosserat model)

$$m(\boldsymbol{U}, \boldsymbol{V}) = \sum_{i=1}^{n_{\mathcal{E}}} \rho^i A^i \int_0^{\ell^i} \tilde{\boldsymbol{u}}^i \cdot \tilde{\boldsymbol{v}}^i \, dx_1,$$

Problem (EvoP)

Find $\boldsymbol{U} \in L^2(0, T; V_{\mathrm{stent}})$ *such that*

$$\partial_{{\color{red}tt}} m(\boldsymbol{U}, \boldsymbol{V}) + a(\boldsymbol{U}, \boldsymbol{V}) = f(\boldsymbol{V}), \qquad \boldsymbol{V} \in V_{\mathrm{stent}}.$$

Eigenvalue problem

Problem (EigP)

Find $(\lambda, \boldsymbol{U}) \in \mathbb{R} \times V_{\mathrm{stent}}$, $\boldsymbol{U} \neq 0$ *such that*

$$a(\boldsymbol{U}, \boldsymbol{V}) = \lambda^2 m(\boldsymbol{U}, \boldsymbol{V}), \qquad \boldsymbol{V} \in V_{\mathrm{stent}}.$$

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Problem (EigQ)

Find $(\lambda, (\boldsymbol{U}, \boldsymbol{\Xi})) \in \mathbb{R} \times (H^1(\mathcal{N}; \mathbb{R}^6) \times L^2(\mathcal{N}; \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3)$, $(\boldsymbol{U}, \boldsymbol{\Xi}) \neq 0$ *such that*

$$\begin{aligned} a(\boldsymbol{U}, \boldsymbol{V}) + b(\boldsymbol{V}, \boldsymbol{\Xi}) &= \lambda^2 m(\boldsymbol{U}, \boldsymbol{V}), \qquad \boldsymbol{V} \in H^1(\mathcal{N}; \mathbb{R}^6), \\ b(\boldsymbol{U}, \boldsymbol{\Theta}) &= 0, \qquad \boldsymbol{\Theta} \in L^2(\mathcal{N}; \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3. \end{aligned}$$

$\Rightarrow$ continuous $\inf - \sup$ condition guarantees that the resolvent set is nonempty

Convergence theory by Boffi–Brezzi–Marini

Recall solution operators $T : \mathcal{H} \to \mathrm{Dom}(a)$ and $S : \mathcal{H} \to \mathrm{Dom}(B)$,

$$\begin{aligned} &a(T\boldsymbol{f}, \boldsymbol{V}) + b(S\boldsymbol{f}, \boldsymbol{V}) = (\boldsymbol{f}, \boldsymbol{V}), \qquad \boldsymbol{V} \in \mathrm{Dom}(a), \\ &b(\boldsymbol{\Theta}, T\boldsymbol{f}) = 0, \qquad \boldsymbol{\Theta} \in \mathrm{Dom}(B^*). \end{aligned} \tag{1}$$

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$\Rightarrow$ Bounded compact operators T_h norm converge to T.

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$\Rrightarrow$ If the resolvent is converging somewhere – say at $z = 0$ –then it converges for every z in resolvent set $\rho(T) = \mathbb{C} \setminus \mathrm{Spec}(T)$.

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$\Rrightarrow$ Analogous definition of T_h.

Convergence rate – comments

⇛ Let λ, u and λ_h and u_h be eigenvalues and eigenvectors from V^h

⇛ Then

$$|\lambda - \lambda_h| \leq C\|T - T_h\|_{\mathrm{Dom}(a)} = O(I - \mathcal{I}^2) = O(h^2)$$
$$\|\boldsymbol{U} - \boldsymbol{U}_h\|_{\mathrm{Dom}(a)} \leq C\|T - T_h\|_{\mathrm{Dom}(a)} = O(I - \mathcal{I}^2) = O(h^2)\ .$$

eigenfuction c.rate optimal, eigenvalue c.rate not.

⇛ If S_h exists, then

$$|\lambda - \lambda_h| \leq C\|T - T_h\|_{\mathrm{Dom}(a)}\|S - S_h\|\ .$$

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⇛ λ_h is not a Ritz value of the solution operator T.
⇛ Note that here we also have the "singular mass" operator, and so we are actually studying the convergence of $T_h M_h$!

Structure recapitulation

$\Rrightarrow$ Jordan structure as a consequence of algebraic constraints.

Lemma

Consider (EigQ) and let $(\mathbf{E}, \mathbf{K})$ *be its block operator matrix representation. Then there exists a nonsingular* $\mathbf{V}$ *with the property that*

$$\hat{\mathsf{E}} = \mathbf{V}^*\mathbf{E}\mathbf{V} = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \mathbf{M} & \\ & & & & 0 \end{bmatrix}, \quad \mathbf{V}^{-1}\boldsymbol{z} = \begin{bmatrix} \hat{\boldsymbol{z}}_1 \\ \hat{\boldsymbol{z}}_2 \\ \hat{\boldsymbol{z}}_3 \\ \hat{\boldsymbol{z}}_4 \\ \hat{\boldsymbol{z}}_5 \end{bmatrix}$$

$$\hat{\mathsf{K}} = \mathbf{V}^*\mathbf{K}\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & \hat{\mathbf{B}}_{51}^T \\ 0 & \hat{\mathbf{A}}_{22} & 0 & \hat{\mathbf{B}}_{42}^T & 0 \\ 0 & 0 & \hat{\mathbf{A}}_{33} & 0 & 0 \\ 0 & \hat{\mathbf{B}}_{42} & 0 & 0 & 0 \\ \hat{\mathbf{B}}_{51} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{V}^*\boldsymbol{F} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hat{\boldsymbol{F}}_4 \\ 0 \end{bmatrix}$$

where $\hat{\mathbf{A}}_{33} = \hat{\mathbf{A}}_{33}^T$, $\hat{\mathbf{B}}_{42}$, *and* $\hat{\mathbf{B}}_{51}$ *are invertible, and* $\hat{\boldsymbol{F}}_4 = \boldsymbol{F}_3$.

Recapitulation

- Sobolev spaces on graphs – nice review by O. Post
- Good interpolation operators for lower order spaces – hard because of the contact conditions in junctions! Geometry of the graph plays a role.
- for second order problems see Arioli and Benzi 2015.

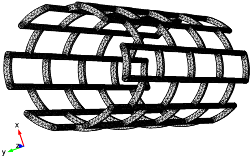
The interplay of geometry and constraints

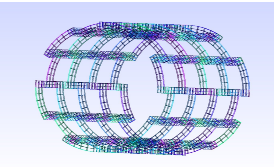
Doing interpolation on each edge and then assembling into a graph sometimes fails.

Verification of the 1D model

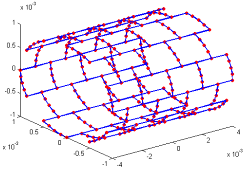
Compare 1D and 3D solutions (jointly with K. Schmidt and A. Semin)

- geometry



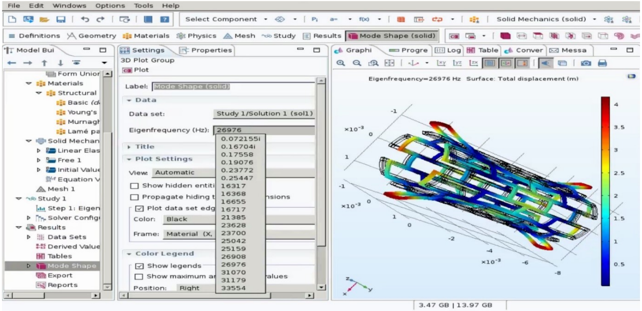


- discretization



Verification of the 1D model – COMSOL, CONCEPTS

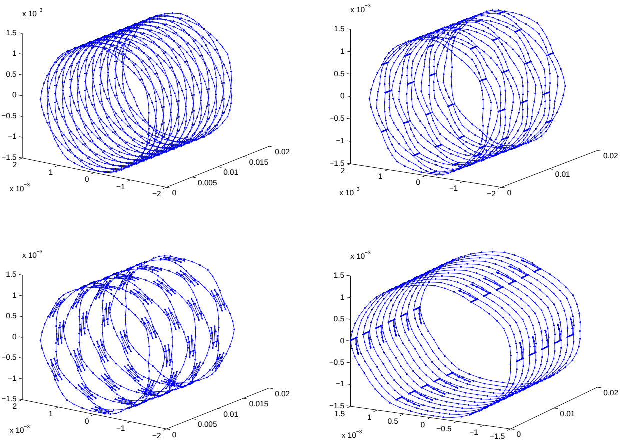
- similar phenomena



- Problems with thin geometries
  - switch to compiled code in CONCEPTS (K.Schmidt)
- Error between 1D reduced model and CONCEPTS 3D model is **2%**.

Examples of eigenproblem for 1D stent model

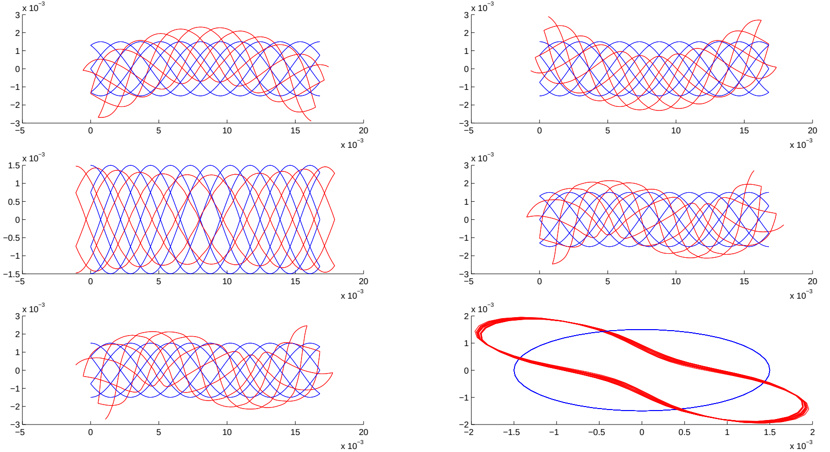
Four stent meshes considered
(similar to Palmaz, Express; Cypher and Xience by Cordis)



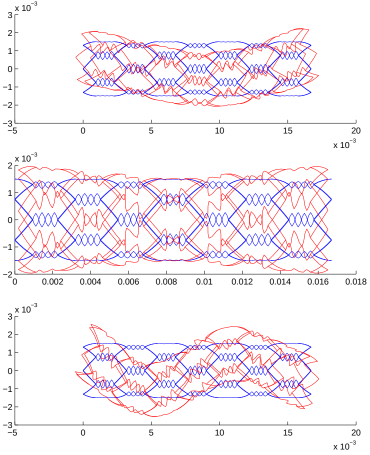
Leading eigenvalues

	Palmaz	Cypher	Express	Xience
1.	1.033	0.8894	0.06014	0.05488
2.	1.033	0.8895	0.06014	0.05488
3.	5.265	1.3683	0.32504	0.28767
4.	7.499	3.5328	0.33972	0.32201
5.	7.499	3.5329	0.33973	0.32201
6.	11.329	3.6604	0.58740	0.58038

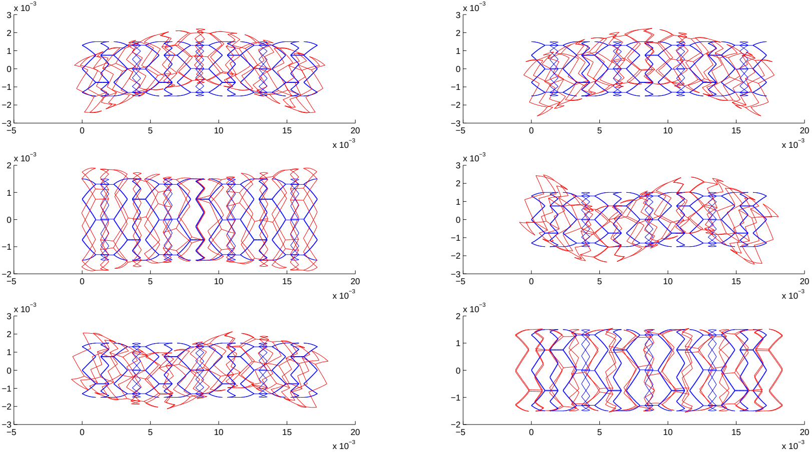
Palmaz



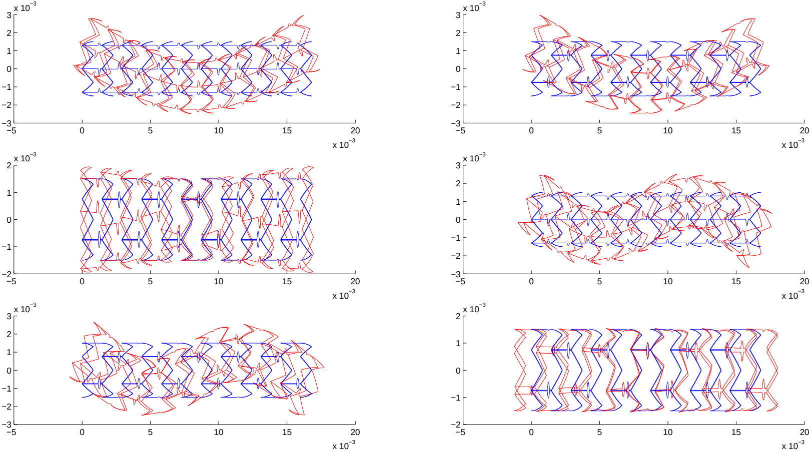
Cypher


x 10^-3
3
2
1
0
−1
−2
−3
−5
0
5
10
15
20
x 10^-3
2
1
0
−1
−2
0
0.002
0.004
0.006
0.008
0.01
0.012
0.014
0.016
0.018
1.5
0.5
−0.5
−1.5
−4
−3
4

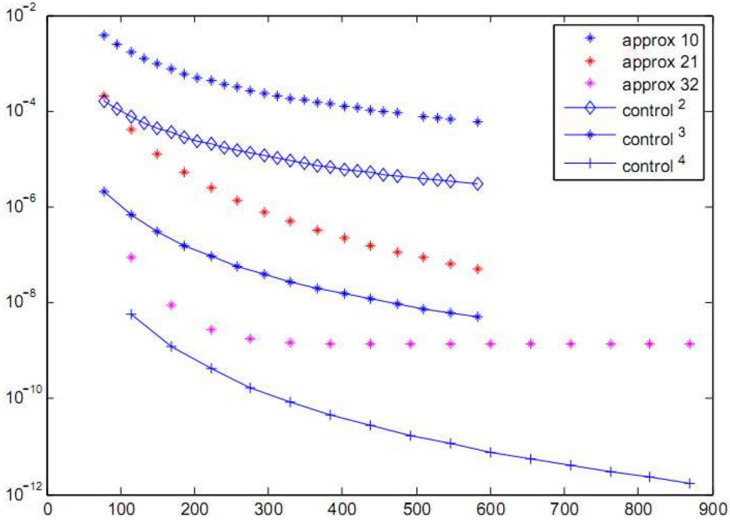
Express



Xience



Convergence rates



Evolution problems

Structure of the pencil

- 0 is in the resolvent set
- Problem is index 2 (∞ eigenvalue has Jordan chain of length 2)
- Imaginary eigenvalues are semisimple
- use deflation and exponential integrators, or backward differentiation schemes like BDF-2, or Volker's favorite Radau2a.

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$\Rrightarrow$ Perhaps another time in more detail :)

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The time stepper actually reads

⇛ In the phase space implicit midpoint rule reads

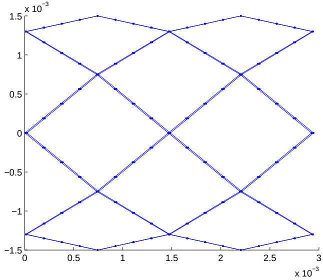
$$\begin{bmatrix} E & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} u_{k+1} - u_k \\ v_{k+1} - v_k \end{bmatrix} = \begin{bmatrix} 0 & K \\ -K & 0 \end{bmatrix} \begin{bmatrix} u_{k+1} + u_k \\ v_{k+1} + v_k \end{bmatrix} \frac{h}{2} + \begin{bmatrix} f(\cdot) \\ 0 \end{bmatrix} h$$

⇛ Port Hamiltonian formulation since K is invertible

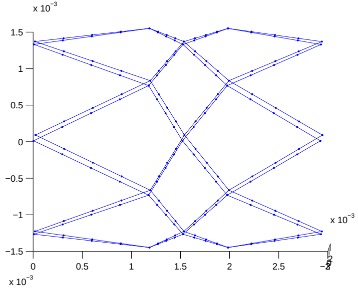
⇛ Time stepper in action

Optimal design of stents

Minimization of maximal radial displacement – minimize the discrepancy to the artery without stenosis.



Original configuration



`fminsearch` "Optimal" configuration

Thank you for your attention!

Acknowledgment

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