Quantum Graph Properties via Pseudo Orbits and Lyndon Words

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Outline

1. Lyndon word decompositions
2. q-nary graphs
3. Pseudo orbit approach

Lyndon words

A word on an alphabet of q letters is a *Lyndon word* if it is strictly smaller in lexicographic order than all its cyclic shifts.

Example: binary Lyndon words length ≤ 3,

$$0 <_{\mathrm{lex}} 001 <_{\mathrm{lex}} 01 <_{\mathrm{lex}} 011 <_{\mathrm{lex}} 1 \ .$$

The standard decomposition

Theorem 1 (Chen, Fox, Lyndon)

Every word w can be uniquely written as a concatenation of Lyndon words in non-increasing lexicographic order, the standard decomposition of w.

Example: standard decompositions of binary words length 3,

$$\begin{array}{cccc} (0)(0)(0) & (01)(0) & (1)(0)(0) & (1)(1)(0) \\ (001) & (011) & (1)(01) & (1)(1)(1) \end{array}$$

A standard decomposition $w = v_1 v_2 \ldots v_k$ with v_j a Lyndon word and $v_j \geq_{\mathrm{lex}} v_{j+1}$ is *strictly decreasing* if $v_j >_{\mathrm{lex}} v_{j+1}$.

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Binary words length 4 and 5.

0000	0100	1000	1100
0001	0101	1001	1101
0010	0110	1010	1110
0011	0111	1011	1111

00000	01000	10000	11000
00001	01001	10001	11001
00010	01010	10010	11010
00011	01011	10011	11011
00100	01100	10100	11100
00101	01101	10101	11101
00110	01110	10110	11110
00111	01111	10111	11111

Binary words length 4 and 5.

(0)(0)(0)(0)	(01)(0)(0)	(1)(0)(0)(0)	(1)(1)(0)(0)
(0001)	(01)(01)	**(1)(001)**	(1)(1)(01)
(001)(0)	**(011)(0)**	**(1)(01)(0)**	(1)(1)(1)(0)
(0011)	**(0111)**	**(1)(011)**	(1)(1)(1)(1)

(0)(0)(0)(0)(0)	(01)(0)(0)(0)	(1)(0)(0)(0)(0)	(1)(1)(0)(0)(0)
(00001)	**(01)(001)**	**(1)(0001)**	(1)(1)(001)
(0001)(0)	(01)(01)(0)	**(1)(001)(0)**	(1)(1)(01)(0)
(00011)	**(01011)**	**(1)(0011)**	(1)(1)(011)
(001)(0)(0)	(011)(0)(0)	(1)(01)(0)(0)	(1)(1)(1)(0)(0)
(00101)	**(011)(01)**	(1)(01)(01)	(1)(1)(1)(01)
(0011)(0)	**(0111)(0)**	**(1)(011)(0)**	(1)(1)(1)(1)(0)
(00111)	**(01111)**	**(1)(0111)**	(1)(1)(1)(1)(1)

Theorem 2 (Band, H., Sepanski)

For words of length $n \geq 2$ the no. of strictly decreasing standard decompositions is,

$$(q-1)q^{n-1} \ .$$

- Hence, the proportion of words length n with strictly decreasing standard decompositions is $\frac{q-1}{q}$.
- i.e. half of binary words have strictly decreasing standard decompositions.
- Proof relies on generating functions and a classical result,

$$\sum_{l|m} l L_q(l) = q^m \ . \tag{1}$$

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q-nary graphs

- $V = q^m$ vertices labeled by words length m.
- $E = q^{m+1}$ directed edges e, each labeled by word w length $m+1$.
- *Origin vertex* $o(e)$, first m letters of w.
- *Terminal vertex* $t(e)$, last m letters of w.
- $2q$-regular
- **Spectral gap:** adjacency matrix has simple eigenvalue 1 and eigenvalue 0 with multiplicity $V-1$. (Maximal spectral gap and maximally mixing.)

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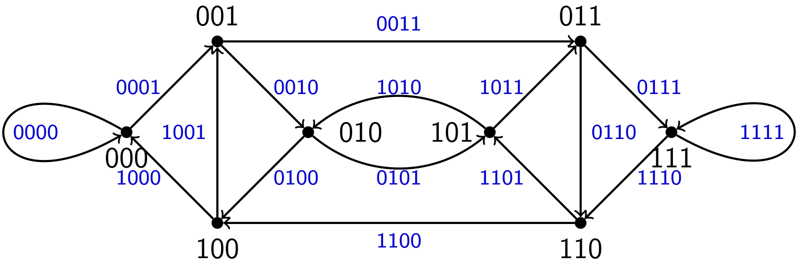
Example: binary graph with 2^3 vertices



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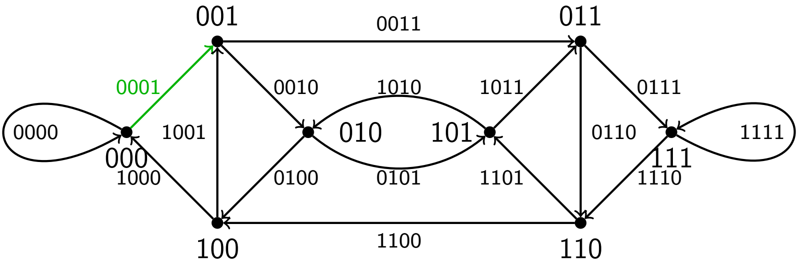

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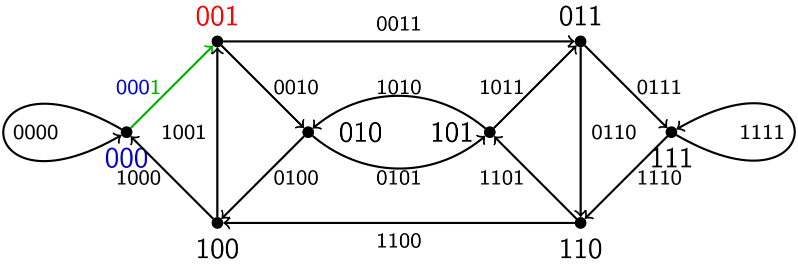

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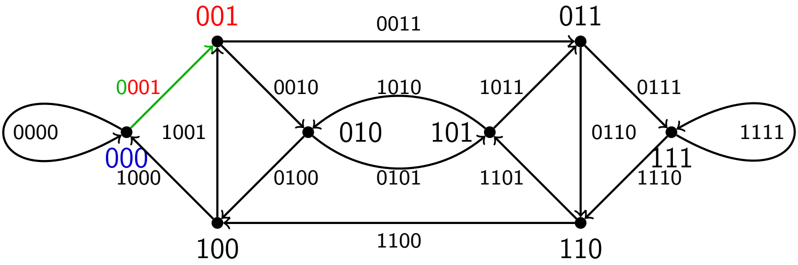

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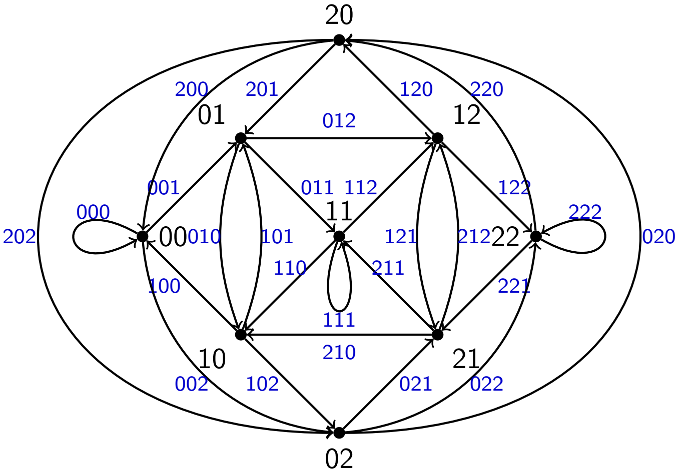
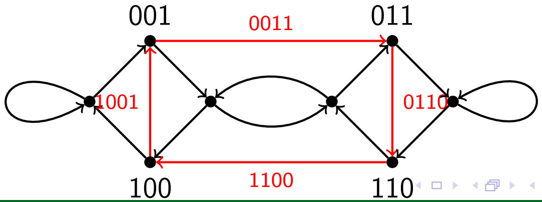
Example: ternary graph with 3^2 vertices



Periodic orbits

- A *path* length l is labeled by a word $w = a_1, \ldots, a_{l+m}$.
- A *closed path* length l is labeled by $w = a_1, \ldots, a_l$.
- A *periodic orbit* γ is the equivalence class of closed paths under cyclic shifts.
- A *primitive periodic orbit* is a periodic orbit that is not a repartition of a shorter orbit.
- Primitive periodic orbits length l are in 1-to-1 correspondence with Lyndon words length l.

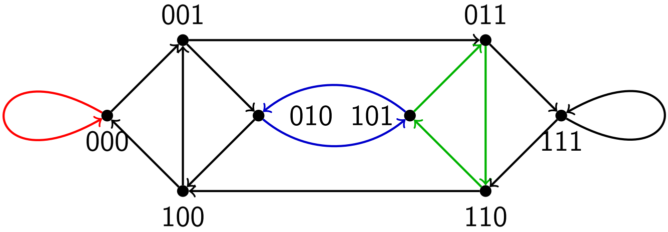
Example: 0011 is a primitive periodic orbit length 4.



Pseudo orbits

- A *pseudo orbit* $\tilde{\gamma} = \{\gamma_1, \ldots, \gamma_M\}$ is a set of periodic orbits.
- A *primitive pseudo orbit* $\bar{\gamma}$ is a set of primitive periodic orbits where no periodic orbit appears more than once.
- **Note:** there is a bijection between primitive pseudo orbits and strictly decreasing standard decompositions.

Example: 011010 has strictly decreasing standard decomposition (011)(01)(0).



Quantum graph

To quantize graph; assign a unitary vertex scattering matrix $\sigma^{(v)}$ to each vertex v.

Example

A democratic choice is the *discrete Fourier transform matrix*,

$$\sigma^{(v)}_{e,e'} = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{d_v-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(d_v-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{q-1} & \omega^{2(q-1)} & \dots & \omega^{(q-1)(q-1)} \end{pmatrix}$$

$\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}}$ a primitive q-th root of unity.

Characteristic polynomial

Combine vertex scattering matrices into an $E \times E$ matrix Σ,

$$\Sigma_{e,e'} = \begin{cases} \sigma^{(v)}_{e,e'} & v = t(e') = o(e) \\ 0 & \text{otherwise} \end{cases}, \qquad (2)$$

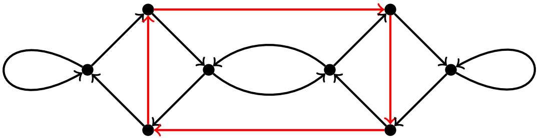
Quantum evolution op. $U(k) = \mathrm{e}^{\mathrm{i}kL}\Sigma$, with $L = \mathrm{diag}\{l_1, \ldots, l_E\}$.

Characteristic polynomial of $U(k)$

$$F_\xi(k) = \det(\xi \mathrm{I} - U(k)) = \sum_{n=0}^{E} a_n \xi^{E-n}$$

- Spectrum corresponds to roots of $F_1(k) = 0$.
- Riemann-Siegel lookalike formula, $a_n = a_E a^*_{E-n}$.

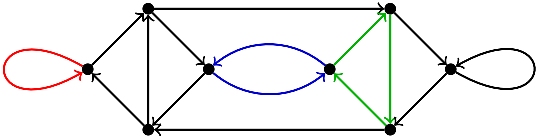
Periodic orbits on a quantum graph



To periodic orbit $\gamma = (e_1, \ldots, e_m)$ on a quantum graph associate,

- *topological length* $E_\gamma = m$.
- *metric length* $l_\gamma = \sum_{e_j \in \gamma} l_{e_j}$.
- *stability amplitude* $A_\gamma = \Sigma_{e_2 e_1} \Sigma_{e_3 e_2} \ldots \Sigma_{e_n e_{n-1}} \Sigma_{e_1 e_m}$.

Pseudo orbits on a quantum graph



To pseudo orbit $\tilde{\gamma} = \{\gamma_1, \ldots, \gamma_M\}$ associate,

- $m_{\tilde{\gamma}} = M$ no. of periodic orbits in $\tilde{\gamma}$.
- *topological length* $E_{\tilde{\gamma}} = \sum_{j=1}^{M} E_{\gamma_j}$.
- *metric length* $l_{\tilde{\gamma}} = \sum_{j=1}^{M} l_{\gamma_j}$.
- *stability amplitude* $A_{\tilde{\gamma}} = \prod_{j=1}^{M} A_{\gamma_j}$.

Theorem 3 (Band,H.,Joyner)

Coefficients of the characteristic polynomial $F_{\xi}(k)$ are given by,

$$a_n = \sum_{\bar{\gamma} |\, E_{\bar{\gamma}} = n} (-1)^{m_{\bar{\gamma}}} A_{\bar{\gamma}} \exp\left(\mathrm{i} k l_{\bar{\gamma}}\right) \ ,$$

where the finite sum is over all primitive pseudo orbits topological length n.

Idea

- Expand $\det\left(\xi \mathrm{I} - U(k)\right)$ as a sum over permutations.
- A permutation $\rho \in S_E$ can contribute iff $\rho(e)$ is connected to e for all e in ρ.
- Representing ρ as a product of disjoint cycles each cycle is a primitive periodic orbit.

Variance of coefficients of the characteristic polynomial

$$\langle a_n \rangle_k = \sum_{\bar{\gamma} |\, E_{\bar{\gamma}} = n} (-1)^{m_{\bar{\gamma}}} A_{\bar{\gamma}} \lim_{K \to \infty} \frac{1}{K} \int_0^K \mathrm{e}^{\mathrm{i}k l_{\bar{\gamma}}} \mathrm{d}k = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \langle |a_n|^2 \rangle_k &= \sum_{\bar{\gamma}, \bar{\gamma}' | E_{\bar{\gamma}} = E_{\bar{\gamma}'} = n} (-1)^{m_{\bar{\gamma}} + m_{\bar{\gamma}'}} A_{\bar{\gamma}} \bar{A}_{\bar{\gamma}'} \lim_{K \to \infty} \frac{1}{K} \int_0^K \mathrm{e}^{\mathrm{i}k(l_{\bar{\gamma}} - l_{\bar{\gamma}'})} \mathrm{d}k \\ &= \sum_{\bar{\gamma}, \bar{\gamma}' | E_{\bar{\gamma}} = E_{\bar{\gamma}'} = n} (-1)^{m_{\bar{\gamma}} + m_{\bar{\gamma}'}} A_{\bar{\gamma}} \bar{A}_{\bar{\gamma}'} \delta_{l_{\bar{\gamma}}, l_{\bar{\gamma}'}} \end{aligned} \tag{3}$$

Diagonal contribution

$$\langle |a_n|^2 \rangle_{\mathrm{diag}} = \sum_{\bar{\gamma} | E_{\bar{\gamma}} = n} |A_{\bar{\gamma}}|^2$$

Background

- Variance of coeffs of the characteristic polynomial of graphs – Kottos and Smilansky (1999).
- Spectral statistics of binary graphs – Tanner (2000)&(2001).
- Variance of coeffs of characteristic polynomial of binary graphs via permanent of transition matrix – Tanner (2002)

Random matrix variance

$$\langle |a_n|^2 \rangle_{\mathrm{COE}} = 1 + \frac{n(E-n)}{E+1}$$
$$\langle |a_n|^2 \rangle_{\mathrm{CUE}} = 1$$

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Diagonal contribution

- Transition probability $|\sigma^{(v)}_{e,e'}|^2 = \dfrac{1}{q}$.
- No. of primitive pseudo orbits length n equal to no. of strictly decreasing standard decompositions of words length n, $(q-1)q^{n-1}$.

Diagonal contribution

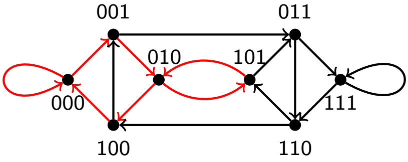
$$\langle |a_n|^2 \rangle_{\mathrm{diag}} = \sum_{\bar{\gamma} | E_{\bar{\gamma}} = n} |A_{\bar{\gamma}}|^2 = \sum_{\bar{\gamma} | E_{\bar{\gamma}} = n} \frac{1}{q^n} = \frac{q-1}{q}$$

For a sequence of graphs with increasing connectivity q the diagonal contribution approaches the random matrix result,

$$\langle |a_n|^2 \rangle_{\mathrm{CUE}} = 1 \ .$$

Off-diagonal contributions (with Tori Hudgins)

Figure of 8 pseudo orbit pairs.
e.g. $\bar{\gamma} = \{0000101\}$, $\bar{\gamma}' = \{00001, 01\}$ have same metric length.



Scattering matrix at intersection vertex $v = 010$,

$$\sigma^{(v)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \tag{4}$$

$A_{\bar{\gamma}}$ and $A_{\bar{\gamma}'}$ differ by -1 but $m_{\bar{\gamma}} = 1$ and $m_{\bar{\gamma}'} = 2$. Hence,

$$(-1)^{m_{\bar{\gamma}} + m_{\bar{\gamma}'}} A_{\bar{\gamma}} \bar{A}_{\bar{\gamma}'} = |A_{\bar{\gamma}}|^2 .$$

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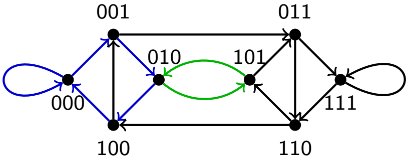

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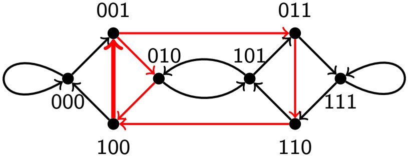
Figure of 8 pseudo orbit pairs with longer encounters.
e.g. $\bar{\gamma} = \{0010011\}$ and $\bar{\gamma}' = \{001, 0011\}$ have same metric length; **both use edge** 1001 **twice**.



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Contributions of figure 8 pairs intersecting at a point and with longer encounters cancel in the limit of long pseudo orbits.

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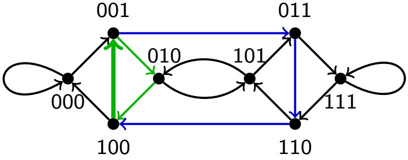

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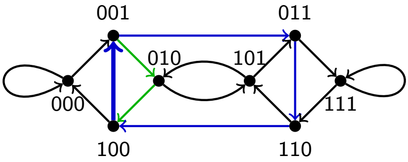

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Summary

- New result for Lyndon words decompositions.
- Graph spectrum encoded in finite number of short primitive pseudo orbits.
- RMT behavior requires stronger conditions.

R. Band, J. M. Harrison and M. Sepanski, "Lyndon word decompositions and pseudo orbits on q-nary graphs," *J. Math. Anal. Appl.* **470** (2019) 135–144 `arXiv:1610:03808`

R. Band, J. M. Harrison and C. H. Joyner, "Finite pseudo orbit expansions for spectral quantities of quantum graphs," *J. Phys. A: Math. Theor.* **45** (2012) 325204 `arXiv:1205.4214`

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