The Landau Hamiltonian coupled with an electric δ-potential

Markus Holzmann

Joint work with J. Behrndt, P. Exner, and V. Lotoreichik
Graz University of Technology

Differential Operators on Graphs and Waveguides, Graz, 2019-03-01

The Landau Hamiltonian

- Let $B > 0$ and $\mathbf{A} := \frac{1}{2}B(-x_2, x_1)^\top$

The Landau Hamiltonian

- Let $B > 0$ and $\mathbf{A} := \frac{1}{2}B(-x_2, x_1)^\top$
- The Landau Hamiltonian acting in $L^2(\mathbb{R}^2)$ is

$$A_0 f := (i\nabla + \mathbf{A})^2 f, \quad \operatorname{dom} A_0 = \{f \in L^2(\mathbb{R}^2) : (i\nabla + \mathbf{A})^2 f \in L^2(\mathbb{R}^2)\}$$

The Landau Hamiltonian

- Let $B > 0$ and $\mathbf{A} := \frac{1}{2}B(-x_2, x_1)^\top$
- The Landau Hamiltonian acting in $L^2(\mathbb{R}^2)$ is

$$A_0 f := (i\nabla + \mathbf{A})^2 f, \quad \operatorname{dom} A_0 = \{f \in L^2(\mathbb{R}^2) : (i\nabla + \mathbf{A})^2 f \in L^2(\mathbb{R}^2)\}$$

- A_0 describes the propagation of a quantum particle in $\mathbb{R}^2$ in a magnetic field of strength $\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1 = B$

The Landau Hamiltonian

- Let $B > 0$ and $\mathbf{A} := \frac{1}{2}B(-x_2, x_1)^\top$
- The Landau Hamiltonian acting in $L^2(\mathbb{R}^2)$ is

$$A_0 f := (i\nabla + \mathbf{A})^2 f, \quad \operatorname{dom} A_0 = \{f \in L^2(\mathbb{R}^2) : (i\nabla + \mathbf{A})^2 f \in L^2(\mathbb{R}^2)\}$$

- A_0 describes the propagation of a quantum particle in $\mathbb{R}^2$ in a magnetic field of strength $\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1 = B$
- Properties of A_0:

The Landau Hamiltonian

- Let $B > 0$ and $\mathbf{A} := \frac{1}{2}B(-x_2, x_1)^\top$
- The Landau Hamiltonian acting in $L^2(\mathbb{R}^2)$ is

$$A_0 f := (i\nabla + \mathbf{A})^2 f, \quad \operatorname{dom} A_0 = \{f \in L^2(\mathbb{R}^2) : (i\nabla + \mathbf{A})^2 f \in L^2(\mathbb{R}^2)\}$$

- A_0 describes the propagation of a quantum particle in $\mathbb{R}^2$ in a magnetic field of strength $\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1 = B$
- Properties of A_0:
  - $A_0 = A_0^*$

The Landau Hamiltonian

- Let $B > 0$ and $\mathbf{A} := \frac{1}{2}B(-x_2, x_1)^\top$
- The Landau Hamiltonian acting in $L^2(\mathbb{R}^2)$ is

$$A_0 f := (i\nabla + \mathbf{A})^2 f, \quad \operatorname{dom} A_0 = \{f \in L^2(\mathbb{R}^2) : (i\nabla + \mathbf{A})^2 f \in L^2(\mathbb{R}^2)\}$$

- A_0 describes the propagation of a quantum particle in $\mathbb{R}^2$ in a magnetic field of strength $\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1 = B$
- Properties of A_0:
  - $A_0 = A_0^*$
  - $\sigma(A_0) = \sigma_{\text{ess}}(A_0) = \{B(2q+1) : q \in \mathbb{N}_0\}$

The Landau Hamiltonian

- Let $B > 0$ and $\mathbf{A} := \frac{1}{2}B(-x_2, x_1)^\top$
- The Landau Hamiltonian acting in $L^2(\mathbb{R}^2)$ is

$$A_0 f := (i\nabla + \mathbf{A})^2 f, \quad \operatorname{dom} A_0 = \{f \in L^2(\mathbb{R}^2) : (i\nabla + \mathbf{A})^2 f \in L^2(\mathbb{R}^2)\}$$

- A_0 describes the propagation of a quantum particle in $\mathbb{R}^2$ in a magnetic field of strength $\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1 = B$
- Properties of A_0:
  - $A_0 = A_0^*$
  - $\sigma(A_0) = \sigma_{\text{ess}}(A_0) = \{B(2q+1) : q \in \mathbb{N}_0\}$
  - $\Lambda_q := B(2q+1)$ is an eigenvalue of infinite multiplicity and called Landau level

Statement of the problem

- Relatively compact perturbations of A_0 smear the Landau levels to eigenvalue clusters:

Statement of the problem

- Relatively compact perturbations of A_0 smear the Landau levels to eigenvalue clusters:
  - $(i\nabla + \mathbf{A})^2 + V$ with sign definite and compactly supported V: Raikov, Rozenblum, Pushnitski, ...

Statement of the problem

- Relatively compact perturbations of A_0 smear the Landau levels to eigenvalue clusters:
  - $(i\nabla + \mathbf{A})^2 + V$ with sign definite and compactly supported V: Raikov, Rozenblum, Pushnitski, ...
  - $(i\nabla + \mathbf{A})^2$ on an unbounded domain Ω with suitable boundary conditions: Bruneau, Goffeng, Kachmar, Miranda, Persson Sundqvist, Pushnitski, Rozenblum, ...

Statement of the problem

- Relatively compact perturbations of A_0 smear the Landau levels to eigenvalue clusters:
  - $(i\nabla + \mathbf{A})^2 + V$ with sign definite and compactly supported V: Raikov, Rozenblum, Pushnitski, ...
  - $(i\nabla + \mathbf{A})^2$ on an unbounded domain Ω with suitable boundary conditions: Bruneau, Goffeng, Kachmar, Miranda, Persson Sundqvist, Pushnitski, Rozenblum, ...
- Goal: similar result for $A_\alpha := (i\nabla + \mathbf{A})^2 + \alpha\delta_\Sigma$

Statement of the problem

- Relatively compact perturbations of A_0 smear the Landau levels to eigenvalue clusters:
  - $(i\nabla + \mathbf{A})^2 + V$ with sign definite and compactly supported V: Raikov, Rozenblum, Pushnitski, ...
  - $(i\nabla + \mathbf{A})^2$ on an unbounded domain Ω with suitable boundary conditions: Bruneau, Goffeng, Kachmar, Miranda, Persson Sundqvist, Pushnitski, Rozenblum, ...
- Goal: similar result for $A_\alpha := (i\nabla + \mathbf{A})^2 + \alpha\delta_\Sigma$
  - Conditions on $\alpha : \Sigma \to \mathbb{R}$ which ensure the clustering of the eigenvalues

Statement of the problem

- Relatively compact perturbations of A_0 smear the Landau levels to eigenvalue clusters:
  - $(i\nabla + \mathbf{A})^2 + V$ with sign definite and compactly supported V: Raikov, Rozenblum, Pushnitski, ...
  - $(i\nabla + \mathbf{A})^2$ on an unbounded domain Ω with suitable boundary conditions: Bruneau, Goffeng, Kachmar, Miranda, Persson Sundqvist, Pushnitski, Rozenblum, ...
- Goal: similar result for $A_\alpha := (i\nabla + \mathbf{A})^2 + \alpha\delta_\Sigma$
  - Conditions on $\alpha : \Sigma \to \mathbb{R}$ which ensure the clustering of the eigenvalues
  - accumulation rate of the eigenvalues in terms of α

Definition of the operator

- Assumptions:
  - $\Sigma \subset \mathbb{R}^2$ is a closed $C^{1,1}$-curve

Definition of the operator

- Assumptions:
  - $\Sigma \subset \mathbb{R}^2$ is a closed $C^{1,1}$-curve
  - $\alpha \in L^\infty(\Sigma)$ is real valued

Definition of the operator

- Assumptions:
  - $\Sigma \subset \mathbb{R}^2$ is a closed $C^{1,1}$-curve
  - $\alpha \in L^\infty(\Sigma)$ is real valued
- Define

$$\mathfrak{a}_\alpha[f] := \big\|(i\nabla + \mathbf{A})f\big\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + \int_\Sigma \alpha \big|f|_\Sigma\big|^2 \mathrm{d}\sigma,$$

$$\operatorname{dom}\mathfrak{a}_\alpha := \big\{f \in L^2(\mathbb{R}^2) : (i\nabla + \mathbf{A})f \in L^2(\mathbb{R}^2;\mathbb{C}^2)\big\}.$$

Definition of the operator

- Assumptions:
  - $\Sigma \subset \mathbb{R}^2$ is a closed $C^{1,1}$-curve
  - $\alpha \in L^\infty(\Sigma)$ is real valued
- Define

$$\mathfrak{a}_\alpha[f] := \big\|(i\nabla + \mathbf{A})f\big\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + \int_\Sigma \alpha \big|f|_\Sigma\big|^2 \mathrm{d}\sigma,$$

$$\mathrm{dom}\, \mathfrak{a}_\alpha := \big\{f \in L^2(\mathbb{R}^2) : (i\nabla + \mathbf{A})f \in L^2(\mathbb{R}^2;\mathbb{C}^2)\big\}.$$

- $\mathfrak{a}_\alpha$ is symmetric, closed, semibounded from below

Definition of the operator

- Assumptions:
  - $\Sigma \subset \mathbb{R}^2$ is a closed $C^{1,1}$-curve
  - $\alpha \in L^\infty(\Sigma)$ is real valued
- Define

$$\mathfrak{a}_\alpha[f] := \big\|(i\nabla + \mathbf{A})f\big\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + \int_\Sigma \alpha \big|f|_\Sigma\big|^2 \mathrm{d}\sigma,$$

$$\operatorname{dom}\mathfrak{a}_\alpha := \big\{f \in L^2(\mathbb{R}^2) : (i\nabla + \mathbf{A})f \in L^2(\mathbb{R}^2;\mathbb{C}^2)\big\}.$$

- $\mathfrak{a}_\alpha$ is symmetric, closed, semibounded from below
- Associated self-adjoint operator is A_α

Definition of the operator

- Assumptions:
  - $\Sigma \subset \mathbb{R}^2$ is a closed $C^{1,1}$-curve
  - $\alpha \in L^\infty(\Sigma)$ is real valued
- Define

$$\mathfrak{a}_\alpha[f] := \left\|(i\nabla + \mathbf{A})f\right\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + \int_\Sigma \alpha \left|f|_\Sigma\right|^2 \mathrm{d}\sigma,$$

$$\operatorname{dom}\mathfrak{a}_\alpha := \left\{f \in L^2(\mathbb{R}^2) : (i\nabla + \mathbf{A})f \in L^2(\mathbb{R}^2;\mathbb{C}^2)\right\}.$$

- $\mathfrak{a}_\alpha$ is symmetric, closed, semibounded from below
- Associated self-adjoint operator is A_α

Theorem (Behrndt, Exner, MH, Lotoreichik)

There exist real valued $V_\varepsilon \in L^\infty(\mathbb{R}^2)$ with $V_\varepsilon \to \alpha\delta_\Sigma$ in the distributional sense, such that

$$(i\nabla + \mathbf{A})^2 + V_\varepsilon \to A_\alpha$$

in the norm resolvent sense.

A Krein type resolvent formula for A_α

- Let $\lambda \in \rho(A_0)$ and G_λ be the Green function for A_0, i.e.

$$(A_0 - \lambda)^{-1} f(x) = \int_{\mathbb{R}^2} G_\lambda(x, y) f(y) \mathrm{d}y$$

A Krein type resolvent formula for A_α

- Let $\lambda \in \rho(A_0)$ and G_λ be the Green function for A_0, i.e.

$$(A_0 - \lambda)^{-1} f(x) = \int_{\mathbb{R}^2} G_\lambda(x, y) f(y) \mathrm{d}y$$

- Define $\gamma(\lambda) : L^2(\Sigma) \to L^2(\mathbb{R}^2)$,

$$\gamma(\lambda)\varphi(x) = \int_\Sigma G_\lambda(x, y)\varphi(y) \mathrm{d}\sigma(y), \quad x \in \mathbb{R}^2$$

Markus Holzmann,
Differential Operators on Graphs and Waveguides, Graz, 2019-03-01

A Krein type resolvent formula for A_α

- Let $\lambda \in \rho(A_0)$ and G_λ be the Green function for A_0, i.e.

$$(A_0 - \lambda)^{-1} f(x) = \int_{\mathbb{R}^2} G_\lambda(x, y) f(y) \mathrm{d}y$$

- Define $\gamma(\lambda) : L^2(\Sigma) \to L^2(\mathbb{R}^2)$,

$$\gamma(\lambda)\varphi(x) = \int_\Sigma G_\lambda(x, y)\varphi(y) \mathrm{d}\sigma(y), \quad x \in \mathbb{R}^2$$

- Define $M(\lambda) : L^2(\Sigma) \to L^2(\Sigma)$,

$$M(\lambda)\varphi(x) = \int_\Sigma G_\lambda(x, y)\varphi(y) \mathrm{d}\sigma(y), \quad x \in \Sigma$$

A Krein type resolvent formula for A_α

- Let $\lambda \in \rho(A_0)$ and G_λ be the Green function for A_0, i.e.

$$(A_0 - \lambda)^{-1} f(x) = \int_{\mathbb{R}^2} G_\lambda(x, y) f(y) \mathrm{d}y$$

- Define $\gamma(\lambda) : L^2(\Sigma) \to L^2(\mathbb{R}^2)$,

$$\gamma(\lambda)\varphi(x) = \int_\Sigma G_\lambda(x, y)\varphi(y)\mathrm{d}\sigma(y), \quad x \in \mathbb{R}^2$$

- Define $M(\lambda) : L^2(\Sigma) \to L^2(\Sigma)$,

$$M(\lambda)\varphi(x) = \int_\Sigma G_\lambda(x, y)\varphi(y)\mathrm{d}\sigma(y), \quad x \in \Sigma$$

Proposition (Ožanová; Behrndt, Exner, MH, Lotoreichik)

$$(A_\alpha - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1} \alpha \gamma(\bar{\lambda})^*$$

A Krein type resolvent formula for A_α

Proposition

$$(A_\alpha - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\bar{\lambda})^*$$

Consequences:

A Krein type resolvent formula for A_α

Proposition

$$(A_\alpha - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\bar{\lambda})^*$$

Consequences:

- $(A_\alpha - \lambda)^{-1} - (A_0 - \lambda)^{-1}$ is compact

A Krein type resolvent formula for A_α

Proposition

$$(A_\alpha - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\bar{\lambda})^*$$

Consequences:

- $(A_\alpha - \lambda)^{-1} - (A_0 - \lambda)^{-1}$ is compact
- $\sigma_{\text{ess}}(A_\alpha) = \sigma_{\text{ess}}(A_0) = \{\Lambda_q = B(2q+1) : q \in \mathbb{N}_0\}$

A Krein type resolvent formula for A_α

Proposition

$$(A_\alpha - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\bar{\lambda})^*$$

Consequences:

- $(A_\alpha - \lambda)^{-1} - (A_0 - \lambda)^{-1}$ is compact
- $\sigma_{\text{ess}}(A_\alpha) = \sigma_{\text{ess}}(A_0) = \{\Lambda_q = B(2q+1) : q \in \mathbb{N}_0\}$
- Goal: analyze $\sigma(A_\alpha)$ in the neighborhoods of Λ_q

A Krein type resolvent formula for A_α

Proposition

$$(A_\alpha - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\bar{\lambda})^*$$

Consequences:

- $(A_\alpha - \lambda)^{-1} - (A_0 - \lambda)^{-1}$ is compact
- $\sigma_{\text{ess}}(A_\alpha) = \sigma_{\text{ess}}(A_0) = \{\Lambda_q = B(2q + 1) : q \in \mathbb{N}_0\}$
- Goal: analyze $\sigma(A_\alpha)$ in the neighborhoods of Λ_q
- Strategy:
    - Apply abstract perturbation theory to $(A_\alpha - \lambda)^{-1}$ and $(A_0 - \lambda)^{-1}$

A Krein type resolvent formula for A_α

Proposition

$$(A_\alpha - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\bar{\lambda})^*$$

Consequences:

- $(A_\alpha - \lambda)^{-1} - (A_0 - \lambda)^{-1}$ is compact
- $\sigma_{\text{ess}}(A_\alpha) = \sigma_{\text{ess}}(A_0) = \{\Lambda_q = B(2q+1) : q \in \mathbb{N}_0\}$
- Goal: analyze $\sigma(A_\alpha)$ in the neighborhoods of Λ_q
- Strategy:
  - Apply abstract perturbation theory to $(A_\alpha - \lambda)^{-1}$ and $(A_0 - \lambda)^{-1}$
  - Relate $-\gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\bar{\lambda})^*$ to certain Toeplitz type operators

A Krein type resolvent formula for A_α

Proposition

$$(A_\alpha - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\bar{\lambda})^*$$

Consequences:

- $(A_\alpha - \lambda)^{-1} - (A_0 - \lambda)^{-1}$ is compact
- $\sigma_{\text{ess}}(A_\alpha) = \sigma_{\text{ess}}(A_0) = \{\Lambda_q = B(2q+1) : q \in \mathbb{N}_0\}$
- Goal: analyze $\sigma(A_\alpha)$ in the neighborhoods of Λ_q
- Strategy:
  - Apply abstract perturbation theory to $(A_\alpha - \lambda)^{-1}$ and $(A_0 - \lambda)^{-1}$
  - Relate $-\gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\bar{\lambda})^*$ to certain Toeplitz type operators
  - Use knowledge on Toeplitz operators

Some results from perturbation theory

- Assumptions:
  - $T = T^*$

Some results from perturbation theory

- Assumptions:
  - $T = T^*$
  - $\Lambda \in \sigma_{\mathsf{p}}(T)$ isolated with $\dim \ker(T - \Lambda) = \infty$

Some results from perturbation theory

- Assumptions:
  - $T = T^*$
  - $\Lambda \in \sigma_{\mathrm{p}}(T)$ isolated with $\dim \ker(T - \Lambda) = \infty$
  - $P_\Lambda =$ projection onto $\ker(T - \Lambda)$

Some results from perturbation theory

- Assumptions:
  - $T = T^*$
  - $\Lambda \in \sigma_p(T)$ isolated with $\dim \ker(T - \Lambda) = \infty$
  - P_Λ = projection onto $\ker(T - \Lambda)$
  - $W = W^*$ compact with $\sigma(W) = \{\mu_k^+ : k \in \mathbb{N}\} \cup \{-\mu_k^- : k \in \mathbb{N}\}$ and $\mu_1^\pm \geq \mu_2^\pm \geq \mu_3^\pm \geq \cdots \geq 0$

Some results from perturbation theory

- Assumptions:
  - $T = T^*$
  - $\Lambda \in \sigma_p(T)$ isolated with $\dim \ker(T - \Lambda) = \infty$
  - $P_\Lambda =$ projection onto $\ker(T - \Lambda)$
  - $W = W^*$ compact with $\sigma(W) = \{\mu_k^+ : k \in \mathbb{N}\} \cup \{-\mu_k^- : k \in \mathbb{N}\}$ and $\mu_1^\pm \geq \mu_2^\pm \geq \mu_3^\pm \geq \cdots \geq 0$
- Question: Eigenvalues $\lambda_k^\pm$ of $T + W$ in $(\Lambda - \tau, \Lambda]$ and $[\Lambda, \Lambda + \tau)$

Some results from perturbation theory

- Assumptions:
  - $T = T^*$
  - $\Lambda \in \sigma_p(T)$ isolated with $\dim \ker(T - \Lambda) = \infty$
  - $P_\Lambda =$ projection onto $\ker(T - \Lambda)$
  - $W = W^*$ compact with $\sigma(W) = \{\mu_k^+ : k \in \mathbb{N}\} \cup \{-\mu_k^- : k \in \mathbb{N}\}$ and $\mu_1^\pm \geq \mu_2^\pm \geq \mu_3^\pm \geq \dots \geq 0$
- Question: Eigenvalues $\lambda_k^\pm$ of $T + W$ in $(\Lambda - \tau, \Lambda]$ and $[\Lambda, \Lambda + \tau)$

Proposition (Pushnitski, Rozenblum)

If $W \geq 0$ *and* $\dim \operatorname{rank} P_\Lambda W P_\Lambda = \infty$, *then* $\forall \varepsilon > 0 \exists l \in \mathbb{N}$ *s.t.*

$$(1 - \varepsilon)\mu_{k+l}^+ \leq \lambda_k^+ - \Lambda \leq (1 + \varepsilon)\mu_{k-l}^+.$$

Some results from perturbation theory

- Assumptions:
  - $T = T^*$
  - $\Lambda \in \sigma_p(T)$ isolated with $\dim \ker(T - \Lambda) = \infty$
  - $P_\Lambda =$ projection onto $\ker(T - \Lambda)$
  - $W = W^*$ compact with $\sigma(W) = \{\mu_k^+ : k \in \mathbb{N}\} \cup \{-\mu_k^- : k \in \mathbb{N}\}$ and $\mu_1^\pm \geq \mu_2^\pm \geq \mu_3^\pm \geq \cdots \geq 0$
- Question: Eigenvalues $\lambda_k^\pm$ of $T + W$ in $(\Lambda - \tau, \Lambda]$ and $[\Lambda, \Lambda + \tau)$

Proposition (Pushnitski, Rozenblum)

If $W \leq 0$ and $\dim \operatorname{rank} P_\Lambda W P_\Lambda = \infty$, then $\forall \varepsilon > 0 \exists l \in \mathbb{N}$ s.t.

$$(1 - \varepsilon)\mu_{k+l}^- \leq \Lambda - \lambda_k^- \leq (1 + \varepsilon)\mu_{k-l}^-.$$

Some results from perturbation theory

- Assumptions:
  - $T = T^*$
  - $\Lambda \in \sigma_p(T)$ isolated with $\dim \ker(T - \Lambda) = \infty$
  - $P_\Lambda =$ projection onto $\ker(T - \Lambda)$
  - $W = W^*$ compact with $\sigma(W) = \{\mu_k^+ : k \in \mathbb{N}\} \cup \{-\mu_k^- : k \in \mathbb{N}\}$ and $\mu_1^\pm \geq \mu_2^\pm \geq \mu_3^\pm \geq \cdots \geq 0$
- Question: Eigenvalues $\lambda_k^\pm$ of $T + W$ in $(\Lambda - \tau, \Lambda]$ and $[\Lambda, \Lambda + \tau)$

Proposition (Behrndt, Exner, MH, Lotoreichik)

If W is not sign-definite, then $\forall \varepsilon > 0 \exists l \in \mathbb{N}$ s.t.

$$\lambda_k^+ - \Lambda \leq (1 + \varepsilon)\mu_{k-l}^+, \quad \Lambda - \lambda_k^- \leq (1 + \varepsilon)\mu_{k-l}^-.$$

A Toeplitz type operator on $\Gamma \subset \Sigma$

- For a fixed $q \in \mathbb{N}_0$ let P_q be the orthogonal projection onto $\ker(A_0 - \Lambda_q)$

A Toeplitz type operator on $\Gamma \subset \Sigma$

- For a fixed $q \in \mathbb{N}_0$ let P_q be the orthogonal projection onto $\ker(A_0 - \Lambda_q)$
- Let $\Gamma \subset \Sigma$ be closed

A Toeplitz type operator on $\Gamma \subset \Sigma$

- For a fixed $q \in \mathbb{N}_0$ let P_q be the orthogonal projection onto $\ker(A_0 - \Lambda_q)$
- Let $\Gamma \subset \Sigma$ be closed
- Define in $L^2(\mathbb{R}^2)$ the bounded sesquilinear form

$$\mathfrak{t}_q^\Gamma[f] := \int_\Gamma \left| (P_q f)|_\Gamma \right|^2 \mathrm{d}\sigma$$

A Toeplitz type operator on $\Gamma \subset \Sigma$

- For a fixed $q \in \mathbb{N}_0$ let P_q be the orthogonal projection onto $\ker(A_0 - \Lambda_q)$
- Let $\Gamma \subset \Sigma$ be closed
- Define in $L^2(\mathbb{R}^2)$ the bounded sesquilinear form

$$\mathfrak{t}_q^\Gamma[f] := \int_\Gamma \left|(P_q f)|_\Gamma\right|^2 \mathrm{d}\sigma$$

- Associated self adjoint operator T_q^Γ

Markus Holzmann,

A Toeplitz type operator on $\Gamma \subset \Sigma$

- For a fixed $q \in \mathbb{N}_0$ let P_q be the orthogonal projection onto $\ker(A_0 - \Lambda_q)$
- Let $\Gamma \subset \Sigma$ be closed
- Define in $L^2(\mathbb{R}^2)$ the bounded sesquilinear form

$$\mathfrak{t}_q^\Gamma[f] := \int_\Gamma \left|(P_q f)|_\Gamma\right|^2 \mathrm{d}\sigma$$

- Associated self adjoint operator T_q^Γ
- Logarithmic energy of a measure μ:

$$I(\mu) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} \mathrm{d}\mu(x) \mathrm{d}\mu(y).$$

A Toeplitz type operator on $\Gamma \subset \Sigma$

- For a fixed $q \in \mathbb{N}_0$ let P_q be the orthogonal projection onto $\ker(A_0 - \Lambda_q)$
- Let $\Gamma \subset \Sigma$ be closed
- Define in $L^2(\mathbb{R}^2)$ the bounded sesquilinear form

$$\mathfrak{t}_q^\Gamma[f] := \int_\Gamma \left| (P_q f)|_\Gamma \right|^2 \mathrm{d}\sigma$$

- Associated self adjoint operator T_q^Γ
- Logarithmic energy of a measure μ:

$$I(\mu) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} \mathrm{d}\mu(x) \mathrm{d}\mu(y).$$

- Logarithmic capacity of a compact set $\mathcal{K}$:

$$\mathrm{Cap}(\mathcal{K}) := \sup \Big\{ e^{-I(\mu)} : \mu \geq 0 \text{ measure on } \mathbb{R}^2, \\ \operatorname{supp} \mu \subset \mathcal{K}, \mu(\mathcal{K}) = 1 \Big\}.$$

Spectrum of the Toeplitz type operator

- Define in $L^2(\mathbb{R}^2)$ and $\Gamma \subset \Sigma$ the bounded sesquilinear form

$$\mathfrak{t}_q^\Gamma[f] := \int_\Gamma \left|(P_q f)|_\Gamma\right|^2 \mathrm{d}\sigma$$

- Associated self-adjoint operator T_q^Γ
- Logarithmic capacity of a compact set $\mathcal{K}$:

$$\text{Cap}(\mathcal{K}) := \sup \Big\{ e^{-I(\mu)} : \mu \geq 0 \text{ measure on } \mathbb{R}^2, \\ \text{supp } \mu \subset \mathcal{K}, \mu(\mathcal{K}) = 1 \Big\}.$$

Spectrum of the Toeplitz type operator

- Define in $L^2(\mathbb{R}^2)$ and $\Gamma \subset \Sigma$ the bounded sesquilinear form

$$\mathfrak{t}_q^\Gamma[f] := \int_\Gamma \left|(P_q f)|_\Gamma\right|^2 \mathrm{d}\sigma$$

- Associated self-adjoint operator T_q^Γ
- Logarithmic capacity of a compact set $\mathcal{K}$:

$$\mathrm{Cap}(\mathcal{K}) := \sup \Big\{ e^{-I(\mu)} : \mu \geq 0 \text{ measure on } \mathbb{R}^2, \\ \operatorname{supp} \mu \subset \mathcal{K}, \mu(\mathcal{K}) = 1 \Big\}.$$

Proposition (Pushnitski, Rozenblum)

(i) $\limsup_{k\to\infty} \left(k!\, s_k(T_q^\Gamma)\right)^{1/k} \leq \frac{B}{2}\left(\mathrm{Cap}\,(\Gamma)\right)^2$.

Spectrum of the Toeplitz type operator

- Define in $L^2(\mathbb{R}^2)$ and $\Gamma \subset \Sigma$ the bounded sesquilinear form

$$\mathfrak{t}_q^\Gamma[f] := \int_\Gamma \left|(P_q f)|_\Gamma\right|^2 \mathrm{d}\sigma$$

- Associated self-adjoint operator T_q^Γ
- Logarithmic capacity of a compact set $\mathcal{K}$:

$$\mathrm{Cap}(\mathcal{K}) := \sup\Big\{e^{-I(\mu)} : \mu \geq 0 \text{ measure on } \mathbb{R}^2, \\ \mathrm{supp}\,\mu \subset \mathcal{K}, \mu(\mathcal{K}) = 1\Big\}.$$

Proposition (Pushnitski, Rozenblum)

(i) $\limsup_{k\to\infty} \left(k!\, s_k(T_q^\Gamma)\right)^{1/k} \leq \frac{B}{2}\left(\mathrm{Cap}\,(\Gamma)\right)^2$.

(ii) *If Γ is a C^∞-smooth arc with two endpoints, then* $\limsup_{k\to\infty} \left(k!\, s_k(T_q^\Gamma)\right)^{1/k} = \frac{B}{2}\left(\mathrm{Cap}\,(\Gamma)\right)^2$.

One sided estimates

- Let $I_q^- := (\Lambda_q - B, \Lambda_q)$ and $I_q^+ := (\Lambda_q, \Lambda_q + B]$

One sided estimates

- Let $I_q^- := (\Lambda_q - B, \Lambda_q)$ and $I_q^+ := (\Lambda_q, \Lambda_q + B]$
- Denote by $\lambda_k^\pm(q)$ the eigenvalues of A_α in $I_q^\pm$

One sided estimates

- Let $I_q^- := (\Lambda_q - B, \Lambda_q)$ and $I_q^+ := (\Lambda_q, \Lambda_q + B]$
- Denote by $\lambda_k^\pm(q)$ the eigenvalues of A_α in $I_q^\pm$

Theorem (Behrndt, Exner, MH, Lotoreichik)

Let $\Gamma := \operatorname{supp} \alpha$. *Then*

$$\limsup_{k\to\infty} \left(k!|\lambda_k^\pm(q) - \Lambda_q|\right)^{1/k} \leq \frac{B}{2} \left(\mathrm{Cap}(\Gamma)\right)^2 .$$

One sided estimates

- Let $I_q^- := (\Lambda_q - B, \Lambda_q)$ and $I_q^+ := (\Lambda_q, \Lambda_q + B]$
- Denote by $\lambda_k^\pm(q)$ the eigenvalues of A_α in $I_q^\pm$

Theorem (Behrndt, Exner, MH, Lotoreichik)

Let $\Gamma := \operatorname{supp}\alpha$. *Then*

$$\limsup_{k\to\infty} \big(k!|\lambda_k^\pm(q) - \Lambda_q|\big)^{1/k} \leq \frac{B}{2}\,(\mathrm{Cap}(\Gamma))^2 .$$

Proof:

- For $\lambda < 0$ sufficiently negative

$$\begin{aligned} &\big|P_q\big((A_\alpha - \lambda)^{-1} - (A_0 - \lambda)^{-1}\big)P_q\big| \\ &\quad = \big| - P_q\gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\lambda)^* P_q\big| \leq cT_q^\Gamma \end{aligned}$$

One sided estimates

- Let $I_q^- := (\Lambda_q - B, \Lambda_q)$ and $I_q^+ := (\Lambda_q, \Lambda_q + B]$
- Denote by $\lambda_k^\pm(q)$ the eigenvalues of A_α in $I_q^\pm$

Theorem (Behrndt, Exner, MH, Lotoreichik)

Let $\Gamma := \operatorname{supp} \alpha$. *Then*

$$\limsup_{k\to\infty} \left(k! |\lambda_k^\pm(q) - \Lambda_q|\right)^{1/k} \leq \frac{B}{2} \left(\operatorname{Cap}(\Gamma)\right)^2.$$

Proof:

- For $\lambda < 0$ sufficiently negative

$$\begin{aligned} &\left| P_q \left((A_\alpha - \lambda)^{-1} - (A_0 - \lambda)^{-1} \right) P_q \right| \\ &\qquad = \left| - P_q \gamma(\lambda) \left(1 + \alpha M(\lambda)\right)^{-1} \alpha \gamma(\lambda)^* P_q \right| \leq c T_q^\Gamma \end{aligned}$$

- Apply result from Pushnitski, Rozenblum

Clustering of eigenvalues from one side

Proposition (Behrndt, Exner, MH, Lotoreichik)

If $\alpha \geq 0$ on Σ, then for all sufficiently small $\varepsilon > 0$ there are only finitely many eigenvalues of A_α in $(\Lambda_q - \varepsilon, \Lambda_q)$.

Clustering of eigenvalues from one side

Proposition (Behrndt, Exner, MH, Lotoreichik)

If $\alpha \geq 0$ on Σ, then for all sufficiently small $\varepsilon > 0$ there are only finitely many eigenvalues of A_α in $(\Lambda_q - \varepsilon, \Lambda_q)$.

Proposition (Behrndt, Exner, MH, Lotoreichik)

If $\alpha > 0$ on Σ, then infinitely many eigenvalues of A_α accumulate to Λ_q from above.

Clustering of eigenvalues from one side

Proposition (Behrndt, Exner, MH, Lotoreichik)

If $\alpha \geq 0$ on Σ, then for all sufficiently small $\varepsilon > 0$ there are only finitely many eigenvalues of A_α in $(\Lambda_q - \varepsilon, \Lambda_q)$.

Proposition (Behrndt, Exner, MH, Lotoreichik)

If $\alpha > 0$ on Σ, then infinitely many eigenvalues of A_α accumulate to Λ_q from above.

Proof:

Clustering of eigenvalues from one side

Proposition (Behrndt, Exner, MH, Lotoreichik)

If $\alpha \geq 0$ on Σ, then for all sufficiently small $\varepsilon > 0$ there are only finitely many eigenvalues of A_α in $(\Lambda_q - \varepsilon, \Lambda_q)$.

Proposition (Behrndt, Exner, MH, Lotoreichik)

If $\alpha > 0$ on Σ, then infinitely many eigenvalues of A_α accumulate to Λ_q from above.

Proof:

- For $\lambda < 0$ sufficiently negative and $\alpha > 0$,

$$P_q\big((A_\alpha - \lambda)^{-1} - (A_0 - \lambda)^{-1}\big)P_q = -P_q\gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\lambda)^* P_q$$

has infinite dimensional rank

Clustering of eigenvalues from one side

Proposition (Behrndt, Exner, MH, Lotoreichik)

If $\alpha \geq 0$ on Σ, then for all sufficiently small $\varepsilon > 0$ there are only finitely many eigenvalues of A_α in $(\Lambda_q - \varepsilon, \Lambda_q)$.

Proposition (Behrndt, Exner, MH, Lotoreichik)

If $\alpha > 0$ on Σ, then infinitely many eigenvalues of A_α accumulate to Λ_q from above.

Proof:

- For $\lambda < 0$ sufficiently negative and $\alpha > 0$,

  $$P_q\big((A_\alpha-\lambda)^{-1}-(A_0-\lambda)^{-1}\big)P_q = -P_q\gamma(\lambda)\big(1+\alpha M(\lambda)\big)^{-1}\alpha\gamma(\lambda)^* P_q$$

  has infinite dimensional rank
- Apply result from Pushnitski, Rozenblum

Some results from perturbation theory

- Assumptions:
  - $T = T^*$
  - $\Lambda \in \sigma_p(T)$ isolated with $\dim \ker(T - \Lambda) = \infty$
  - $P_\Lambda =$ projection onto $\ker(T - \Lambda)$
  - $W = W^*$ compact with
    $\sigma(W) = \{\mu_k^+ : k \in \mathbb{N}\} \cup \{-\mu_k^- : k \in \mathbb{N}\}$ and
    $\mu_1^\pm \geq \mu_2^\pm \geq \mu_3^\pm \geq \cdots \geq 0$
- Question: Eigenvalues $\lambda_k^\pm$ of $T + W$ in $(\Lambda - \tau, \Lambda]$ and $[\Lambda, \Lambda + \tau)$

Proposition (Pushnitski, Rozenblum)

If $W \leq 0$ *and* $\dim \operatorname{rank} P_\Lambda W P_\Lambda = \infty$, *then* $\forall \varepsilon > 0 \exists l \in \mathbb{N}$ *s.t.*

$$(1 - \varepsilon)\mu_{k+l}^- \leq \Lambda - \lambda_k^- \leq (1 + \varepsilon)\mu_{k-l}^-.$$

Clustering of eigenvalues from one side

Proposition (Behrndt, Exner, MH, Lotoreichik)

If $\alpha \leq 0$ on Σ, then for all sufficiently small $\varepsilon > 0$ there are only finitely many eigenvalues of A_α in $(\Lambda_q, \Lambda_q + \varepsilon)$.

Clustering of eigenvalues from one side

Proposition (Behrndt, Exner, MH, Lotoreichik)

If $\alpha \leq 0$ on Σ, then for all sufficiently small $\varepsilon > 0$ there are only finitely many eigenvalues of A_α in $(\Lambda_q, \Lambda_q + \varepsilon)$.

Proposition (Behrndt, Exner, MH, Lotoreichik)

If $\alpha < 0$ on Σ, then then infinitely many eigenvalues of A_α accumulate to Λ_q from below.

Two sided estimates

- Let $I_q^- := (\Lambda_q - B, \Lambda_q)$ and $I_q^+ := (\Lambda_q, \Lambda_q + B]$

Two sided estimates

- Let $I_q^- := (\Lambda_q - B, \Lambda_q)$ and $I_q^+ := (\Lambda_q, \Lambda_q + B]$
- Denote by $\lambda_k^\pm(q)$ the eigenvalues of A_α in $I_q^\pm$

Two sided estimates

- Let $I_q^- := (\Lambda_q - B, \Lambda_q)$ and $I_q^+ := (\Lambda_q, \Lambda_q + B]$
- Denote by $\lambda_k^\pm(q)$ the eigenvalues of A_α in $I_q^\pm$

Theorem (Behrndt, Exner, MH, Lotoreichik)

Let $\alpha \geq 0$ and assume that $\Gamma := \operatorname{supp} \alpha$ be a C^∞-smooth curve with two endpoints. Then

$$\limsup_{k \to \infty} \left(k!(\lambda_k^+(q) - \Lambda_q) \right)^{1/k} = \frac{B}{2} (\operatorname{Cap}(\Gamma))^2 .$$

Two sided estimates

- Let $I_q^- := (\Lambda_q - B, \Lambda_q)$ and $I_q^+ := (\Lambda_q, \Lambda_q + B]$
- Denote by $\lambda_k^\pm(q)$ the eigenvalues of A_α in $I_q^\pm$

Theorem (Behrndt, Exner, MH, Lotoreichik)

Let $\alpha \geq 0$ and assume that $\Gamma := \operatorname{supp} \alpha$ be a C^∞-smooth curve with two endpoints. Then

$$\limsup_{k\to\infty} \left(k!(\lambda_k^+(q) - \Lambda_q)\right)^{1/k} = \frac{B}{2} (\mathrm{Cap}(\Gamma))^2 .$$

Theorem (Behrndt, Exner, MH, Lotoreichik)

Let $\alpha \leq 0$ and assume that $\Gamma := \operatorname{supp} \alpha$ be a C^∞-smooth curve with two endpoints. Then

$$\limsup_{k\to\infty} \left(k!(\Lambda_q - \lambda_k^-(q))\right)^{1/k} = \frac{B}{2} (\mathrm{Cap}(\Gamma))^2 .$$

Thank you for your attention!